AFOSR 5485 (5) 952 100 (1) 63-3-6

AD No. 406242

FILE COPY

406 242

LETTERS SCIENCE ART

MONOGRAPH IN MATHEMATICS

ASYMPTOTIC VALUES OF HOLOMORPHIC FUNCTIONS

G. R. MacLANE

This research was supported by the
Mathematics Division, AFOSR,
SRMM
under Contract/Grant ~~AF 18(600)-1135~~
~~AF 47(638)-205~~
AF-AFOSR 62-94

VOL. 49, NO. 1 WINTER 1963

8.10

RICE UNIVERSITY STUDIES

JOURNAL OF WILLIAM MARSH RICE UNIVERSITY

RICE UNIVERSITY STUDIES, successor to the RICE INSTITUTE PAMPHLET, is a quarterly journal of writings in all scholarly disciplines by staff members and other persons associated with Rice University.

EDITOR: Edward Norbeck   ASSISTANT EDITOR: Raemond Craig

INFORMATION FOR AUTHORS

Manuscripts submitted for publication should be addressed to the Editor, Rice University Studies, Rice University, Houston 1, Texas. When preparing manuscripts contributors in the humanities and social sciences are requested to follow the MLA Style Sheet; contributors in science and mathematics are requested to follow the established procedures of major journals in their special fields. Notes and references will appear at the end of the manuscript.

Application to mail at second class postage rate is pending at Houston, Texas.

(4) $5.10 (5) 952 100

RICE UNIVERSITY STUDIES

(7) NA
(8) NA
(9) NA
(12) $3/0
(13) NA
(14) NA
(15) NA
(16) NA
(17) NA
(18) See cover
(19) See cover
(20) NA

(2) MONOGRAPH IN MATHEMATICS,

(6) ASYMPTOTIC VALUES OF HOLOMORPHIC FUNCTIONS,

(10) by G. R. MacLANE

PUBLISHED BY
WILLIAM MARSH RICE UNIVERSITY
HOUSTON 1, TEXAS

G. C.

VOL. 49, NO. 1 WINTER (11) 1963

CONTENTS

ASYMPTOTIC VALUES OF HOLOMORPHIC FUNCTIONS[1]

1. Introduction. The purpose of this paper is to derive some results on the asymptotic values of functions $f(z)$ holomorphic in the unit circle. If f is bounded or has bounded characteristic, then the Fatou-Nevanlinna theorem and related results tell the story. We are concerned with a larger class of functions where a consideration of radial limits is completely inadequate; we must consider asymptotic values along arbitrary curves in $\{ |z| < 1 \}$ which end at a point of $\{ |z| = 1 \}$.

In section 2 we summarize the facts about asymptotic tracts which will be needed later.

In section 3 we define three classes of non-constant holomorphic functions in $\{ |z| < 1 \}$. $\mathcal{A}$ is the class of functions having asymptotic values at a dense set on $\{ |z| = 1 \}$. $\mathcal{B}$ is the class of functions such that there is a set of Jordan arcs Γ in $\{ |z| < 1 \}$, each ending at a point of $\{ |z| = 1 \}$, such that the end points are dense on $\{ |z| = 1 \}$ and such that on each Γ either $f \to \infty$ or f is bounded. The class $\mathcal{L}$ is defined as follows: $f \in \mathcal{L}$ if and only if each level set $\{ z \mid |f(z)| = \lambda \}$ "ends at points" of $\{ |z| = 1 \}$. (The precise definition will be found early in section 3.) The central result is Theorem 1: $\mathcal{A} = \mathcal{B} = \mathcal{L}$.

The remaining two theorems of section 3, as well as Theorems 4 through 9 of section 4, give information on the possible nature of the asymptotic tracts of $f \in \mathcal{A}$. In particular, $f \in \mathcal{A}$ may have a tract, with asymptotic value ∞, which is associated with a whole arc $K \subset \{ |z| = 1 \}$. But if so, to each $\zeta \in K$ there exists a curve Γ ending at ζ on which $f \to \infty$. Various useful conditions associated with such tracts are obtained. On the other hand, Theorem 4 shows that a tract associated with a finite asymptotic value corresponds to a point (never an arc) of $\{ |z| = 1 \}$.

Now one of the obvious examples of a function of class $\mathcal{A}$ is the modular function $f(z)$ mapping $\{ |z| < 1 \}$ onto the universal covering of the sphere with $0, 1, \infty$ removed. As is well-known, the asymptotic values $(0, 1, \text{or } \infty)$ occur just at the vertices of the fundamental regions, and those are merely countable in number. But under suitable conditions we can do better than countable dense. Sections 5 and 6 are devoted to proving Theorem 11: that if $f \in \mathcal{A}$

[1]This research was supported by the Air Force Office of Scientific Research.

and γ is an arc of $\{ \mid z \mid = 1 \}$ on which f does not assume the asymptotic value ∞, then f assumes finite asymptotic values on a subset of γ of positive measure. As shown by Example 11 in section 10, the conclusion can *not* be improved to almost everywhere on γ.

Section 7 is devoted to deriving some sufficient conditions for $f \in \mathcal{A}$. The fundamental condition, Theorem 14, is

$$\text{(I)} \qquad \int_0^1 (1-r) \log^+ \mid f(re^{i\theta}) \mid dr < \infty \qquad (\theta \in \Theta)$$

where Θ is dense on $[0, 2\pi]$. A more restrictive condition is

$$\text{(III)} \qquad \int_0^1 (1-r) \log^+ M(r) \; dr < \infty$$

where $M(r)$ is the maximum modulus of f.

Section 8 is devoted to further sufficient conditions for $f \in \mathcal{A}$. The most significant, Theorem 17, is that if f is normal in the sense of Lehto and Virtanen, then $f \in \mathcal{A}$.

Section 9 considers the question of how many distinct asymptotic values may be associated with a single point z_0 of $\{ \mid z \mid = 1 \}$. Here $f \in \mathcal{A}$ doesn't seem to imply anything particular. But if f satisfies (III) then, Theorem 22, f has no more than 5 distinct asymptotic tracts at z_0. Examples 13 and 14 in section 10 show that Theorem 22 is not susceptible to improvement unless more is assumed about f. The proof of Theorem 22 involves Ahlfors' proof of the Denjoy conjecture on the number of asymptotic tracts of an entire function.

Finally, section 10 contains an assortment of examples illustrating and clarifying the results of the preceding sections. In particular, Example 2 shows that there are functions in $\mathcal{A}$ which possess no radial limits.

It should be observed that all the results are for holomorphic functions. The arguments break down in various places for meromorphic functions. In particular, nothing in the nature of Theorem 14 can hold for meromorphic functions, for, as was shown in [14], there exist meromorphic functions without any asymptotic values for which $\iota(r)$ grows arbitrarily slowly.

For future purposes it is convenient to make the following definition. Let $\{ \gamma_n \}$ be a sequence of continuous compact curves in $\{ |z| < 1 \}$ and let γ be an arc $\{ z \mid |z| = 1, \alpha \leqslant \arg z \leqslant \beta \}$.

DEFINITION. *$\gamma_n \to \gamma$ if and only if for each $\varepsilon > 0$ there exists n_0 such that*

$$\left.\begin{array}{c} \gamma_n \subset \{ 1-\varepsilon < |z| < 1 \} \\ |\inf_{\gamma_n} \arg z - \alpha | < \varepsilon, \qquad |\sup_{\gamma_n} \arg z - \beta | < \varepsilon \end{array}\right\} (n > n_0).$$

One of the basic results, Theorem 14, depends on Theorem 13, which is a generalization of the following well-known lemma of Koebe [9]. *Let $f(z)$ be holomorphic and bounded in $\{ |z| < 1 \}$. Let $\gamma_n \to \gamma$ where $\gamma_n \subset \{ |z| < 1 \}$ and γ is an arc of $\{ |z| = 1 \}$. If $\sup_{\gamma_n} |f(z) - a| \to 0$ as $n \to \infty$ then $f(z) \equiv a$.* It should also be pointed out that the first conclusion of Theorem 9, namely $\mu > 0$, is another generalization of Koebe's lemma.

2. Asymptotic Tracts. It is convenient to recall some basic facts about asymptotic tracts, a concept that goes back to Boutroux [4]; see also Iversen [8] and Valiron [22]. We state these facts for the case of a function *holomorphic* in $\{ |z| < 1 \}$ and non-constant; the ideas involved have a much wider applicability.

A tract $\{ D(\varepsilon), a \}$ associated with the finite value a is a set of **non-void** *domains $D(\varepsilon)$, one for each $\varepsilon > 0$, such that*

(2.1) *$D(\varepsilon)$ is a component of the open set* $\{ z \mid |z| < 1, |f(z) - a| < \varepsilon \}$,

(2.2) $0 < \varepsilon_1 < \varepsilon_2 \Rightarrow D(\varepsilon_1) \subset D(\varepsilon_2)$,

(2.3) $\bigcap_{\varepsilon > 0} D(\varepsilon) = \emptyset.$

If a is replaced by ∞ the only change is to replace $|f(z) - a| < \varepsilon$ in (2.1) by $|f(z)| > 1/\varepsilon$.

The point of (2.3) is clear; we don't want $D(\varepsilon)$ collapsing on a point of $\{ |z| < 1 \}$; that would correspond to a value of f, not an asymptotic value.

It is readily seen that

$$K = \cap \overline{D(\varepsilon)} \tag{2.4}$$

is a non-void, connected, closed subset of $\{ \mid z \mid = 1 \}$. We shall call K the *end* of the tract. If K is a point (arc) then the tract will be called a *point-* (*arc-*) *tract.* The tract is a point-tract if and only if diam $D(\varepsilon) \to 0$ as $\varepsilon \to 0$. The tract will be called *global* if and only if K is the whole circumference $\{ \mid z \mid = 1 \}$ and for each arc $\gamma \subset \{ \mid z \mid = 1 \}$ there is a sequence of arcs $\gamma_n \subset D(1/n)$ such that $\gamma_n \to \gamma$. This last condition is important; Theorem 6 is false unless some such condition is included in the definition of global tract. For an example of a non-global tract for which $K = \{ \mid z \mid = 1 \}$, see Example 3 in section 10. As is well-known, there are functions which have more than one global tract; f can tend to ∞ on one spiral and tend to zero on another. There can be more than one arc-tract associated with a given arc; similarly for point-tracts. But we shall see that for the class of functions we shall consider such multiplicity can arise only in the case of point-tracts.

A tract is uniquely determined by $D(\varepsilon_n)$ satisfying (2.1), (2.2), and (2.3) for some sequence $\varepsilon_n \downarrow 0$. The two tracts $\{D_1(\varepsilon), a_1\}$ and $\{ D_2(\varepsilon), a_2 \}$ are distinct if and only if $D_1(\varepsilon) \cap D_2(\varepsilon) = \emptyset$ for ε sufficiently small; if $a_1 \neq a_2$ they are automatically distinct.

Let $\{ D(\varepsilon), a \}$ be a tract and let Γ: $z = \phi(t), 0 \leqslant t < 1$ be a continuous curve in $\{ \mid z \mid < 1) \}$ such that $\phi(t) \in D(\varepsilon)$ for $1 - \delta(\varepsilon) < t < 1$. In this case we shall say that Γ *belongs to* $\{ D(\varepsilon), a \}$. Because of (2.3), $\mid \phi(t) \mid \to 1$ as $t \uparrow 1$. Clearly $f(z) \to a$ as $\mid z \mid \to 1$ on Γ; that is, f has the *asymptotic value* a on Γ. Given the tract, there exist curves Γ belonging to it. Conversely, if f has the asymptotic value a along some curve Γ on which $\mid z \mid \to 1$, then there is precisely one tract $\{ D(\varepsilon), a \}$ to which Γ belongs. A curve Γ belonging to $\{ D(\varepsilon), a \}$ will tend either to a single point of K or to an arc in K. There always will be some Γ which tends to the whole set K; there may or may not be curves Γ belonging to the tract which tend to proper subsets of K.

(2.5) Let $a \neq \infty$ and let $\{D(\varepsilon), a \}$ be a tract. Then $D(\varepsilon)$ is simply-connected (maximum principle). Let Γ_1 and Γ_2 be two Jordan arcs belonging to $\{ D(\varepsilon), a \}$ which tend to single points ζ_1 and ζ_2 of K. Then $\zeta_1 = \zeta_2$. If Γ_1 and Γ_2 are disjoint except that they start at the same point, say $z = 0$, then any curve Γ tending to ζ_1 and lying in the domain bounded by $\Gamma_1 \cup \Gamma_2$ also belongs to $\{ D(\varepsilon), a \}$.

(2.6) Let Γ_1 and Γ_2 be two arcs in $\{\,|z|<1\,\}$ ending at the same point ζ, $|\zeta|=1$, such that $f \to a_i$ on Γ_i, a_i finite. If Γ_1 and Γ_2 belong to different tracts (which is certainly the case if $a_1 \neq a_2$) then there is a curve Γ, between Γ_1 and Γ_2 and ending at ζ, on which $f \to \infty$. "Between" means that Γ_1 and Γ_2 may be extended so that $\Gamma_1 \cup \Gamma_2$ is a Jordan curve and Γ lies in the corresponding Jordan domain.

(2.7) If $a=\infty$ then $D(\varepsilon)$ may have any connectivity and the results in (2.5) all fail. See Example 3 in section 10.

(2.8) Let Γ_1 and Γ_2 be two arcs in $\{\,|z|<1\,\}$ ending at the same point ζ, $|\zeta|=1$, such that $f\to\infty$ on $\Gamma_i (i=1,2)$ but such that Γ_1 and Γ_2 belong to *different* tracts for the value ∞. Then there exists a curve Γ between Γ_1 and Γ_2 and ending at ζ such that f is bounded on Γ.

(2.9) *We shall say that f has the asymptotic value a* **at** ζ, $|\zeta|=1$, *if and only if there is an arc $\Gamma \subset \{\,|z|<1\,\}$ tending to ζ such that $f \to a$ on Γ.*

We conclude this section by remarking, as is well-known, that any holomorphic function f in $\{\,|z|<1\,\}$ has at least one asymptotic tract. The easiest way to see this is as follows: Let $w=f(z)$ map $\{\,|z|<1\,\}$ onto the Riemann surface S over the w-plane. Let the ray R: $w=f(0)+e^{i\alpha}t$, $0\leqslant t\leqslant\infty$ on the w-sphere be chosen so that S has no branch-points over R. (If $f'(0)=0$, use some point $z_1\neq 0$, $f'(z_1)\neq 0$ to start at.) As is easily seen there is a maximal half-open piece of R, $R_\tau : 0\leqslant t<\tau$, $\tau\leqslant\infty$, which may be lifted into S, starting at $f(0)$ in S. The point of the requirement that f be holomorphic is that S has no points over ∞ and hence it is impossible to lift the complete ray R. It is now easily seen that $w_1=f(0)+e^{i\alpha}\tau$ is an asymptotic value of f along the curve in $\{\,|z|<1\,\}$ which corresponds to the lifted R_τ.

Example 3 in section 10 shows that f may have precisely one asymptotic tract.

3. The Classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{L}$.

Let $f(z)$ be holomorphic and non-constant in $\{\,|z|<1\,\}$. For any complex number a ($a=\infty$ permitted) we define the set A_a as follows. *The point ζ belongs to A_a if and only if $|\zeta|=1$ and f has the asymptotic value a at ζ.* See (2.9). If S is any set on the sphere we set

$$A(S) = \bigcup_{a \in S} A_a, \quad A(S) = \emptyset \text{ if } S = \emptyset. \tag{3.1}$$

In particular we set

$$A^* = \bigcup_{a \neq \infty} A_a, \quad A = A^* \cup A_\infty . \tag{3.2}$$

DEFINITION. *$f(z)$ belongs to $\mathcal{A}$ if and only if f is holomorphic and non-constant in $\{ |z| < 1 \}$ and A is dense on $\{ |z| = 1 \}$.*

We define the set B^* as follows. *The point ζ, $|\zeta| = 1$, belongs to B^* if and only if there is an arc Γ in $\{ |z| < 1 \}$, ending at ζ, such that $|f|$ is bounded by some finite constant M on Γ.* In general the bound M will vary as ζ and Γ vary. Also we set

$$B = B^* \cup A_\infty . \tag{3.3}$$

DEFINITION. *$f(z)$ belongs to $\mathcal{B}$ if and only if f is holomorphic and non-constant in $\{ |z| < 1 \}$ and B is dense on $\{ |z| = 1 \}$.*

It is immediately clear that $B^* \supset A^*$, $B \supset A$, and hence

$$\mathcal{A} \subset \mathcal{B}. \tag{3.4}$$

For the following two definitions we distinguish between level set and level curve. Given $f(z)$ in $\{ |z| < 1 \}$, then the *level set* $L(\lambda)$ is $\{ z \mid |f(z)| = \lambda \}$; here λ is any constant $\geqslant 0$. A *level curve* $C(\lambda)$ is any component of $L(\lambda)$.

Now let S be any subset of $\{ |z| < 1 \}$. For each r, $0 < r < 1$, let the components of

$$S \cap \{ r < |z| < 1 \}$$

be $S_i(r)$, $i \in I$. Let $\delta_i(r) = \operatorname{diam} S_i(r)$ and set

$$\delta(r) = \sup_{i \in I} \delta_i(r)$$

with $\delta(r) \equiv 0$ if I is void. Clearly $\delta(r) \downarrow$ as $r \uparrow$. We shall say that S *ends at points of* $\{ |z| = 1 \}$ if and only if $\delta(r) \downarrow 0$ as $r \uparrow 1$.

DEFINITION. *$f(z)$ belongs to the class $\mathcal{L}(\mathcal{L}^*)$ if and only if $f(z)$ is holomorphic and non-constant in $\{ |z| < 1 \}$ and every level set (curve) $L(\lambda)$ $(C(\lambda))$ ends at points of $\{ |z| = 1 \}$.*

It is immediately clear that

$$\mathcal{L} \subset \mathcal{L}^*. \tag{3.5}$$

Remark on the definition of $\mathcal{L}^*$. If $C(\lambda)$ is a level curve of $f \in \mathcal{L}^*$ without double points, then either it is a Jordan curve or each "half" of it ends at one definite point on $\{ \mid z \mid = 1 \}$; *i.e.*, $C(\lambda)$ is a cross-cut of $\{ \mid z \mid < 1 \}$. However, for some λ, $C(\lambda)$ may have a countable infinity of multiple points and for this reason it seems simplest to define $\mathcal{L}^*$ as above in terms of the concept of "ending at points". We remark that *the level curve $C(\lambda)$ ends at points of $\{ \mid z \mid = 1 \}$ if and only if every continuous curve $\Gamma \subset C(\lambda)$ on which $\mid z \mid \to 1$ tends to one definite point ζ, $\mid \zeta \mid = 1$.* It is immediately clear that if $C(\lambda)$ ends at points then every Γ must tend to a point. Suppose now $C(\lambda)$ does not end at points. Since $C(\lambda)$ is connected there exists r_0 such that for $r_0 < r < 1$ every component, $C_i(\lambda,r)$, $i = 1,2, \ldots$ of $C(\lambda)$ in $\{ r < \mid z \mid < 1 \}$ will have a limit point, $z_i(r)$, on $\{ \mid z \mid = r \}$. If for a given r there were infinitely many components $C_i(\lambda,r)$, there would exist a point z_0, $\mid z_0 \mid = r$, every neighborhood of which would meet infinitely many $C_i(\lambda,r)$. But f could not be holomorphic at such a point; thus there are only a finite number of components $C_i(\lambda,r)$, $i = 1,2 \cdots, i(r)$. Pick $\{r_n\}$, $0 < r_n \uparrow 1$. Since $C(\lambda)$ does not end at points, one of the finitely many components $C_i(\lambda,r_1)$ does not end at points, say $C_1(\lambda,r_1)$, and for this component $\delta(r) \downarrow \delta > 0$, where $\delta(r)$ is the $\delta(r)$ used in the definition of ending at points. Then one of the finitely many components of

$$C_1(\lambda,r_1) \cap \{ r_2 < \mid z \mid < 1 \},$$

say $C_1(\lambda,r_2)$, satisfies the same condition $\delta(r) \downarrow \delta > 0$, and so on. Thus we get $C_1(\lambda,r_1) \supset C_1(\lambda,r_2) \supset \cdots$ with diam $C_1(\lambda,r_n) \geqslant \delta$. Let Γ be a curve tending to $\{ \mid z \mid = 1 \}$ and running in $C_1(\lambda,r_1)$, $C_1(\lambda,r_2)$, $\cdots$ in succession; Γ may be chosen so that $C_1(\lambda,r_n)$ contains a compact portion of Γ with diameter $> \delta/2$. Clearly Γ does not tend to one point of $\{ \mid z \mid = 1 \}$ and our remark is proved. It should be noted that this Γ just constructed need not be simple (without double points). It is possible to construct an $f(z)$ which has a level curve $C(1)$ consisting of a sequence of disjoint cross-cuts γ_n of $\{ \mid z \mid < 1 \}$ which tend to an arc γ of $\{ \mid z \mid = 1 \}$ and of a simple arc γ' which interects each γ_n just once and tends to an interior point of γ. Any *simple* $\Gamma \subset C(1)$ will then tend to a point, but $C(1)$ does not end at points.

Our object now is to prove the fundamental Theorem 1.

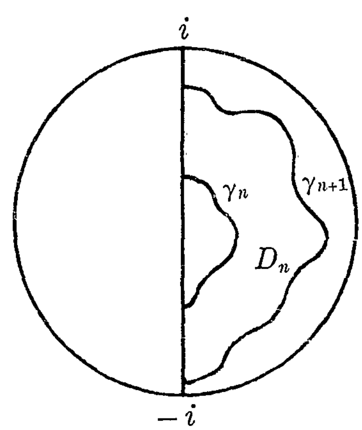
THEOREM 1. *$\mathcal{A} = \mathcal{B} = \mathcal{L} \subset \mathcal{L}^*$, and the inclusion is proper.*

We start with the following lemma.

LEMMA 1. *Let $f(z)$ be holomorphic, non-constant, and bounded in $\{ |z| < 1 \}$. Then $f \in \mathcal{L}$.*

Proof of Lemma 1. Let $|f| < 1$ and suppose $f \notin \mathcal{L}$. It follows then that there exists a λ, $0 < \lambda < 1$, such that the level set $L(\lambda)$ contains a set of disjoint simple arcs γ_n such that the γ_n tend to an arc γ of $\{ |z| = 1 \}$. By means of Fatou's theorem and a linear transformation we may assume that the radial limits $f(\pm i)$ exist and that each γ_n is a cross-cut of the right half unit disc joining a point of the radius $(-i,0)$ to a point of $(0,i)$. Furthermore we may assume that in the right half disc γ_{n+1} separates γ_n from $\gamma = \{ e^{i\theta} \mid |\theta| \leqslant \pi/2 \}$. Let D_n denote the domain bounded by γ_n, γ_{n+1} and two segments of the radii $(0, \pm i)$. Since $|f| = \lambda$ on each γ_n it follows that $|f(\pm i)| = \lambda$. Then from the maximum principle

$$|f(z)| < \lambda + \varepsilon \qquad (z \in D_n, n > n_0(\varepsilon)). \tag{3.6}$$



From (3.6) it follows that

$$\limsup_{z \to \zeta} |f(z)| = \lambda \qquad (\zeta \in \gamma^0) \tag{3.7}$$

where γ^0 is the interior, $|\theta| < \pi/2$, of γ.

We show that the point 1 is not a limit point of zeros of f. Let

$$U_\nu = \{ |z| < 1 \} \cap \{ |z-1| < 1/\nu \}$$

and for some fixed ν let a_ν be a zero of f, $a_\nu \in U_\nu$. The function

$$g_\nu(z) = f(z)\frac{1-\bar{a}_\nu z}{z-a_\nu}$$

is bounded by 1 and also satisfies (3.7). Hence $\log |g_\nu(z)| \leqslant \omega(z) \log \lambda$ where ω is the harmonic measure of γ in the unit disc. Thus

$$\log |f(z)| \leqslant \omega(z) \log \lambda + \log \left| \frac{z-a_\nu}{1-\bar{a}_\nu z} \right| \quad (|z| < 1). \tag{3.8}$$

Let σ_ν denote the segment $\{\arg z = \arg a_\nu, |a_\nu| < |z| < 1\}$. If

$$c = \max \frac{\partial \omega}{\partial r} \qquad (z \in U_1),$$

then c is a finite positive constant since ω is harmonic in the closure of U_1. Then

$$\omega(z) \geqslant 1 - c(1-|z|) \qquad (z \in U_1), \tag{3.9}$$

and hence (3.9) is valid on every σ_ν.

The function

$$h(r,b) = \log \frac{r-b}{1-br} \qquad (0 < b < r < 1),$$

is readily seen to be a concave function of r with $h(1,b) = 0$. Hence $h(r) < h'(1)\ (r-1)$ or

$$\log \frac{r-b}{1-br} < -\frac{1+b}{1-b}(1-r) \quad (0 < b < r < 1).$$

On the segment σ_ν then

$$\log \left| \frac{z-a_\nu}{1-\bar{a}_\nu z} \right| = \log \frac{|z|-|a_\nu|}{1-|\bar{a}_\nu z|} < -\frac{1+|a_\nu|}{1-|a_\nu|}(1-|z|). \tag{3.10}$$

Using (3.9) and (3.10) in (3.8) we get

$$\log |f(z)| \leqslant \log \lambda - \left\{ \frac{1+|a_\nu|}{1-|a_\nu|} + c \log \lambda \right\}(1-|z|), \ (z \in \sigma_\nu). \tag{3.11}$$

Now if 1 were a limit point of zeros of f, then there would be an $a_\nu \in U_\nu$ for each ν. Picking ν large enough so that from the definition of U_ν

$$\frac{1+|a_\nu|}{1-|a_\nu|} > \nu > |c \log \lambda|,$$

we see from (3.11) that $|f(z)| < \lambda$ on σ_ν. But σ_ν is intersected by infinitely many γ_n on which $|f| = \lambda$, and thus we have a contradiction. Hence 1 is not a limit point of zeros of f.

Now let U be one fixed neighborhood U_ν which contains no zeros of f and let $R_k = [0, e^{i\theta_k})$, $k = 1,2$, be two distinct radii which end in U and are such that the radial limits $f(e^{i\theta_k})$ exist. Then

$$|f(e^{i\theta_k})| = \lambda \qquad (k = 1,2). \tag{3.12}$$

Let $\gamma'_n \subset \gamma_n$ be a cross-cut of the sector $\theta_1 < \arg z < \theta_2$ and let G_n be the domain bounded by γ'_n, γ'_{n+1} and two segments of R_1 and R_2. Since $|f| = \lambda$ on γ'_n and because of (3.12), it follows that

$$||f(z)| - \lambda| < \varepsilon$$

on the boundary of G_n for $n > n_0(\varepsilon)$. But for $n > n_1$, $G_n \subset U$ and $f \neq 0$ in G_n. Then it follows from the maximum-minimum principle that

$$||f(z)| - \lambda| < \varepsilon \qquad (z \in G_n, n \geq n_2(\varepsilon)). \tag{3.13}$$

Hence $|f(z)| \to \lambda$ as $z \to \gamma'$, where γ' is the arc of $\{|z| = 1\}$ from θ_1 to θ_2. Then by the reflection principle, the function $\log |f(z)|$, which is harmonic in U, is harmonic in a complete disc about 1. But then $f(z)$ is holomorphic at $z = 1$, which is incompatible with a whole slew of distinct level arcs γ_n coming arbitrarily close to $z = 1$. This completes the proof of Lemma 1.

Proof of Theorem 1. Proof of $\mathcal{B} \subset \mathcal{A}$. Let $f \in \mathcal{B}$. To prove that $f \in \mathcal{A}$ it is clearly enough to prove that if γ is an arc of $\{|z| = 1\}$ such that f does not have the asymptotic value ∞ at any *point* of γ then f has a finite asymptotic value at some point of γ. Since $f \in \mathcal{B}$, there exist two distinct points ζ_1 and ζ_2 on γ and two arcs C_1 and C_2 joining $z = 0$ to ζ_1 and ζ_2, respectively, on which $|f| \leq M$. We may assume that C_1 and C_2 are disjoint except for $z = 0$. Let D denote the domain bounded by C_1, C_2 and the subarc $\gamma' = (\zeta_1, \zeta_2)$ of γ.

If f is bounded in D, then f has radial limits at almost all points of γ' and the proof is complete.

Suppose f is unbounded in D. Then $w = f(z)$ maps D onto a Riemann surface S containing some points over $|w| > M$. Let P_0 be a point of S over w_0, $|w_0| > M$, and let R be a ray from w_0 to ∞ in the w-plane such that R lies in $|w| > M$. R may also be chosen so that it avoids the at most countable set of projections of the branch points of S. Now it is easy to prove that there exists a maximal half-open segment $[w_0,w_1)$ of R which may be lifted into S with starting point P_0. Call the lifted path $\tilde{R}$, and let Γ be the image of $\tilde{R}$ in D. Clearly $f \to w_1$ on Γ and Γ must tend to the boundary of D; that is, f has the asymptotic value w_1 on Γ. But $|w_1| > M$ and $|f| \leqslant M$ on C_1 and C_2. Hence Γ must tend to an arc or point of γ'.

Now if $w_1 = \infty$, it follows from our hypothesis (that f does not have the asymptotic value ∞ at any *point* of γ) that Γ must tend to an arc $\gamma_1 \subset \gamma'$. But this is impossible; for $f \in \mathcal{B}$ and f is bounded on each of a set of curves ending at points which are dense on γ_1. Thus $|w_1| < \infty$.

If now Γ tends to a point of γ', we are done. If Γ tends to a whole arc $\gamma_1 \subset \gamma'$ let ζ_3 and ζ_4 be two distinct interior points of γ_1 and let C_3 and C_4 be arcs ending at ζ_3 and ζ_4 on which f is bounded by M_1. This is possible because of $f \in \mathcal{B}$ and our hypothesis on γ. But $|f| < M_2$ on Γ, and Γ contains arcs, each joining a point of C_3 to a point of C_4, which tend to $\gamma_2 = (\zeta_3, \zeta_4)$. Then by the maximum principle

$$\limsup_{z \to \zeta} |f(z)| \leqslant \max(M_1, M_2) \qquad (\zeta \in \gamma_2).$$

Thus by Fatou's theorem, f has radial limits almost everywhere on γ_2 and f does have a finite asymptotic value at a point of γ. (Actually Γ must tend to a point, for, if not, all the radial limits on γ_2 would be equal to w_1 which, by the Theorem of F. and M. Riesz, would imply $f \equiv w_1$, which is incompatible with $f \in \mathcal{B}$.) Thus we have proved that $\mathcal{B} \subset \mathcal{A}$; together with (3.4) this gives

$$\mathcal{A} = \mathcal{B}. \tag{3.14}$$

An examination of the above proof shows that we have proved something more than we set out to prove. If f is bounded in D, then $A^* \cap \gamma$ has positive measure and hence has the power of the con-

tinuum. On the other hand, if f is unbounded in D, then there is a continuum of permissible rays R, with w_0 and P_0 fixed, leading to distinct finite asymptotic values w_1. Two distinct asymptotic values w_1 will be associated with distinct points of γ because of (2.6). Thus we have proved the following theorem.

THEOREM 2. *Let* $f \in \mathcal{A}$ *and let* γ *be an arc of* $\{ \mid z \mid = 1 \}$ *such that* $A_\infty \cap \gamma = \emptyset$. *Then* $A^* \cap \gamma$ *has the power of the continuum.*

We shall prove in Theorem 11 that $A^* \cap \gamma$ has positive measure, but that will take much more work.

Proof of $\mathcal{B} \subset \mathcal{L}$. Suppose $f \in \mathcal{B}$, $f \notin \mathcal{L}$. Then there is a level set $L(\lambda)$ which does not end at points. That is, each annulus $\{r < \mid z \mid < 1\}$ contains a component of $L(\lambda)$ whose diameter exceeds $2\delta > 0$. From such a component we can pick a simple compact arc whose diameter exceeds δ. By picking a suitable sequence we obtain disjoint arcs γ_n, tending to an arc $\gamma \subset \{ \mid z \mid = 1 \}$, such that $\mid f(z) \mid = \lambda$ on each γ_n. It is impossible then that f have the asymptotic value ∞ at any interior point of γ. But $f \in \mathcal{B}$, and hence there exist two distinct points interior to γ, ζ_1 and ζ_2 such that $\mid f \mid \leqslant M$ on arcs C_1 and C_2 which end at ζ_1 and ζ_2 respectively. We may assume that C_1 and C_2 start at $z = 0$ and have no other point in common. Let D be the domain bounded by C_1, C_2, and the arc $\gamma' = (\zeta_1, \zeta_2) \subset \gamma$. Each γ_n, for $n > n_0$, contains a cross-cut γ'_n of D, joining a point of C_1 to a point of C_2. It follows from the maximum principle that $\mid f(z) \mid < \operatorname{Max}(M, \lambda)$ in D. Map D conformally onto $\{\mid w \mid < 1\}$ and set $f(z) = F(w)$. Clearly F has a level set, containing the images of the γ'_n, which doesn't end at points. That is impossible by Lemma 1 and hence our assumption $f \in \mathcal{B}, f \notin \mathcal{L}$ is impossible. Thus we have proved

$$\mathcal{B} \subset \mathcal{L}. \tag{3.15}$$

Proof of $\mathcal{L} \subset \mathcal{B}$. Let $f \in \mathcal{L}$ and let γ be a given compact arc of $\{\mid z \mid = 1\}$. We shall prove that there is an arc $\Gamma \subset \{\mid z \mid < 1\}$, ending at a point $\zeta \in \gamma$ such that either f is bounded on Γ or $f \to \infty$ on Γ. We shall consider level sets $L(\lambda)$ and level curves $C(\lambda)$. For convenience we shall assume that λ is chosen so that $L(\lambda)$ has no multiple points; that assumption forces the omission of at most a countable number of λ. Then each $C(\lambda)$ is either a Jordan curve or a cross-cut of $\{\mid z \mid < 1\}$, since $f \in \mathcal{L}$.

If there exists, for some λ, a cross-cut $C(\lambda)$ that ends at a point of γ, then we are done. Also, if ζ is a point of γ and there exists a neighborhood of ζ in which f is bounded, then the existence of Γ is trivial.

So we assume that no $C(\lambda)$ has an endpoint on γ and that at each point $\zeta \in \gamma$

$$\limsup_{z \to \zeta} |f(z)| = \infty. \tag{3.16}$$

Consider then a permissible sequence $\lambda_n \uparrow \infty$ and let D_n be that component of $\{z \mid |f(z)| < \lambda_n\}$ which contains $z = 0$; we assume that $|f(0)| < \lambda_1$. Now D_n is simply-connected (maximum principle) and is either the interior of a Jordan curve which is a $C(\lambda_n)$ or is bounded by a finite or countable number of cross-cuts $C(\lambda_n)$ and various points of $\{|z| = 1\}$. In the latter case, because of (3.16) and the fact that no $C(\lambda_n)$ has an end on γ, D_n will have one $C(\lambda_n)$ on its boundary which separates $z = 0$ from γ. Let

$$\gamma = \{ e^{i\theta} \mid \alpha \leqslant \theta \leqslant \beta \}$$

and let R_1 and R_2 be the radii to $e^{i\alpha}$ and $e^{i\beta}$ respectively. Then for either sort of D_n the boundary of D_n will contain an arc γ_n which joins a point of R_1 to a point of R_2 and is a cross-cut of the sector $\alpha < \arg z < \beta$. Now, $|f(z)| = \lambda_n \uparrow \infty$ on γ_n and $\gamma_n \to \gamma$. It follows from Theorem 3 immediately below (which does not depend upon Theorem 1) that f has the asymptotic value ∞ at every single point of γ, which completes the proof of

$$\mathcal{L} \subset \mathcal{B}. \tag{3.17}$$

Now, if we combine (3.5), (3.14), (3.15), and (3.17) we have Theorem 1, except for the propriety of the inclusion. That will be taken care of by Example 16 in section 10.

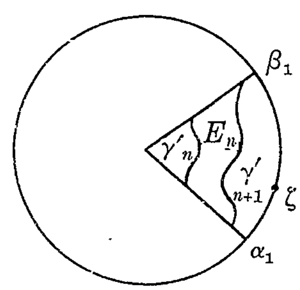
THEOREM 3. *Let $f \in \mathcal{L}$ and let γ_n be a sequence of distinct simple arcs in $\{|z| < 1\}$ which tend to the arc γ of $\{|z| = 1\}$ with the property that*

$$\inf_{z \in \gamma_n} |f(z)| = \mu_n \to \infty \qquad (n \to \infty). \tag{3.18}$$

Then $f(z)$ has an asymptotic tract $\{D(\varepsilon),\infty\}$ with end K such that $\gamma \subset K$ and such that if ζ is an arbitrary point of K then there is a curve Γ belonging to $\{D(\varepsilon),\infty\}$ which ends at ζ. At any interior point ζ of K the only asymptotic values come from this tract $\{D(\varepsilon),\infty\}$.

Remark. If $\{ D(\varepsilon), \infty \}$ is a given tract for an arbitrary holomorphic function for which K is an arc, then we can choose $\gamma_n \subset D(1/n)$ such that $\gamma_n \to K$ and (3.18) is satisfied. So one consequence of Theorem 3 is that any arc-tract $\{D(\varepsilon),\infty\}$ for $f \in \mathcal{L}$ is relatively decent in that one has point asymptotic values on all of K. See Example 3 in section 10. But more than that, Theorem 3 shows that a sequence $\gamma_n \to \gamma$ on which $f \to \infty$ determines *one* tract; that result is false for unrestricted f.

Proof. Let $\gamma = \{ e^{i\theta} \mid \alpha \leqslant \theta \leqslant \beta \}$ and let ζ be an interior point of γ. Choose α_1,β_1, so that $\alpha < \alpha_1 < \arg \zeta < \beta_1 < \beta$. Let $S(\alpha_1,\beta_1)$ denote the sector $\{ \alpha_1 < \arg z < \beta_1, \mid z \mid < 1 \}$ and let $\gamma'_n \subset \gamma_n$ be a cross-cut of $S(\alpha_1,\beta_1)$ joining a point of $\arg z = \alpha_1$ to a point of $\arg z = \beta_1$. By using a subsequence of the γ_n if need be we may assume that each γ_n contains such a cross-cut γ'_n and that γ'_{n+1} separates γ'_n from $\mid z \mid = 1$ within $S(\alpha_1,\beta_1)$. We may also assume that $\mu_n \uparrow \infty$. Let E_n denote the



subdomain of $S(\alpha_1,\beta_1)$ which is bounded by γ'_n, γ'_{n+1}, and two intervals on the boundary radii of $S(\alpha_1,\beta_1)$. Now let $\lambda > 0$ be a given constant and consider the level set $L(\lambda) = \{ z \mid \mid f(z) \mid = \lambda \}$. For $n \geqslant n_o$ let the components of $L(\lambda) \cap E_n$ be $l(n,i)$, $1 \leqslant i \leqslant i_n$. Here we choose $n_o = n_o(\lambda)$ so that by (3.18) no γ_n with $n \geqslant n_o$ intersects $L(\lambda)$. Let

$$\delta_n = \max_i \operatorname{diam} l(n,i) \qquad (n \geqslant n_o). \tag{3.19}$$

Since $f \in \mathcal{L}$

$$\delta_n \to 0 \qquad (n \to \infty). \tag{3.20}$$

For the sake of simplicity we pick λ so that $L(\lambda)$ has no multiple points. Then, by (3.20) for $n \geqslant n_1$, any curve $l(n,i)$ which intersects the radius $R = \{ \arg z = \arg \zeta \}$ will be a Jordan curve contained in E_n. Hence any interval of R, in E_n, on which $|f(z)| < \lambda$ may be replaced by an arc of a level curve $l(n,i)$. Making such replacements (finite in number for any one n) for each $n \geqslant n_1$, we obtain a curve $\Gamma(\lambda)$ such that

$$\liminf_{|z| \to 1,\, z \in \Gamma(\lambda)} |f(z)| \geqslant \lambda.$$

Also $\Gamma(\lambda)$ tends to ζ by (3.20), and $\Gamma(\lambda)$ and R intersect at each intersection of R with any γ'_n.

Now let $\lambda_n \uparrow \infty$ be given. We construct Γ from portions of $\Gamma(\lambda_1)$, $\Gamma(\lambda_2)$, $\cdots$. Let Q_n be the last (max $|z|$) intersection of R with γ'_n, and let $\Gamma(\lambda_k,n)$ be the portion of $\Gamma(\lambda_k)$ joining Q_n to ζ. Follow $\Gamma(\lambda_1)$ from $z = 0$ to Q_{n_1} where n_1 is chosen so that

$$|\arg z - \arg \zeta| < \tfrac{1}{2}, \quad |f(z)| \geqslant \lambda_2 \quad (z \in \Gamma(\lambda_2, n_1)).$$

From Q_{n_1} follow $\Gamma(\lambda_2)$ to the point Q_{n_2} where $n_2 > n_1$ is chosen so that

$$|\arg z - \arg \zeta| < \tfrac{1}{4}, \qquad |f(z)| \geqslant \lambda_3 \quad (z \in \Gamma(\lambda_3, n_2)).$$

Follow $\Gamma(\lambda_3)$ from Q_{n_2} to Q_{n_3} where etc. The composite curve Γ thus constructed clearly tends to ζ and $f(z) \to \infty$ on Γ.

We have assumed so far that ζ is an interior point of γ. Now let ζ be an end point of γ, and let $\{ \zeta_n \}$ be a sequence of interior points of γ with $\zeta_n \to \zeta$. Let Γ_n be a curve to ζ_n on which $f \to \infty$. A curve Γ, tending to ζ, on which $f \to \infty$ is easily constructed by piecing together arcs of the Γ_n's between suitable γ_k's and arcs of the γ_k's. The details of the construction are similar to those of the construction we just finished; so we omit them.

Finally, each asymptotic path Γ to an interior point ζ of γ intersects all the γ_n for $n > n_0$. By (3.18), all γ_n with $n > n_1$ will belong to one and the same domain $D(\varepsilon)$ where $|f(z)| > 1/\varepsilon$. Thus all Γ belong to one fixed tract $\{ D(\varepsilon), \infty \}$. If the end K of this tract contains γ as a *proper* subset, then simply choose arcs $\gamma^*_n \subset D(1/n)$ such that $\gamma^*_n \to \gamma^* = K$. Since γ^*_n and γ^* satisfy the same hypotheses

as γ_n and γ it follows that if $\zeta \in K$ then there is a curve Γ belonging to $\{ D(\varepsilon), \infty \}$ which tends to ζ. The last sentence of Theorem 3 follows, since if Γ ends at ζ, an interior point of K, then Γ must intersect all but a finite number of the γ^*_n. This completes the proof of Theorem 3 and hence also that of Theorem 1.

Clearly $f \in \mathcal{A} \Rightarrow af + b \in \mathcal{A}$ for any constants a and b, $a \neq 0$. Then, by Theorem 1, the similar result obtained by replacing both $\mathcal{A}$'s by $\mathcal{L}$'s holds. That is not immediately clear from the definition of $\mathcal{L}$. One naturally raises the following questions, where $\mathcal{A}^*$ denotes $\mathcal{A}$ plus all constant functions. Is $\mathcal{A}^*$ a linear space? An algebra?

4. **Results on Asymptotic Tracts.** Now that Theorem 1 is proved, *the hypothesis* $f \in \mathcal{L}$ *in Theorem 3 may be replaced by* $f \in \mathcal{A}$ *or* $f \in \mathcal{B}$. Theorems 2 and 3 give two results on the asymptotic values of functions in $\mathcal{A}$. We add the following two almost immediate results.

THEOREM 4. *Let* $f \in \mathcal{A}$ *and let* $\{ D(\varepsilon), a \}$, $a \neq \infty$, *be a tract of* f. *Then* $\{ D(\varepsilon), a \}$ *is a point-tract.*

The proof of this theorem is the argument given in the paragraph immediately preceding (3.14).

THEOREM 5. *Let* $f \in \mathcal{A}$ *and let* B^* *be defined as at the beginning of section 3. Let the open set* $B^{*-\prime}$ *consist of the open arcs* J_n. *Then each* J_n^- *is the end of a single tract* $\{ D(\varepsilon, n), \infty \}$ *of the type in Theorem 3. In particular*

$$B^{*-\prime} \subset \bar{A}_\infty . \tag{4.1}$$

Here closure (bar) and complement (prime) are relative to $\{ |z| = 1 \}$.

Proof. Each J_n satisfies the condition on γ stated just prior to (3.16) in the proof of $\mathcal{L} \subset \mathcal{B}$. That proof and Theorem 3 apply. It is clear from the definition of the J_n that the end of the tract $\{ D(\varepsilon, n), \infty \}$ must be precisely J_n^-. The analog of (4.1) with B^* replaced by A^* is false, as is seen from Example 15 in section 10.

The next three theorems contain further information on arc-tracts.

THEOREM 6. *Let* $f \in \mathcal{A}$. *Then*

(A) *f has a global tract if and only if f is unbounded and all level curves of f are compact.*

(B) *f has a global tract if and only if f is unbounded on every curve Γ in $\{ |z| < 1 \}$ on which $|z| \to 1$.*

Proof. To prove (A), assume first that f has a global tract. Then of course f is unbounded. If $C(\lambda)$ were a non-compact level curve of f, then, since $f \in \mathcal{L}$, $C(\lambda)$ would contain an arc Γ ending at a point $\zeta \in \{ |z| = 1 \}$. But (recalling the definition of global tract in section 2) we can choose γ_n as in Theorem 3, with ζ an interior point of γ. Then Γ would intersect infinitely many γ_n, which is obviously impossible. Now assume that f is unbounded and has compact level curves. Choose any λ for which the level set $L(\lambda)$ has no multiple points and consider $D(\lambda)$, the component of $\{ z \mid |f(z)| < \lambda \}$ which contains $z = 0$; we assume that $\lambda > |f(0)|$. $D(\lambda)$ is not the whole disc $\{ |z| < 1 \}$, since f is unbounded; hence $D(\lambda)$ is the interior of a Jordan curve $J(\lambda)$. Use a sequence $\lambda_n \uparrow \infty$, $D(\lambda_n)$, $J(\lambda_n)$; since $D(\lambda_n) \subset D(\lambda_{n+1})$ and any given disc $\{ |z| < r \}$, $r < 1$, is contained in $D(\lambda_n)$ for sufficiently large n, it follows that the curves $\gamma_n = J(\lambda_n)$ tend to the complete circumference $\{ |z| = 1 \}$. An application of Theorem 3 completes the proof.

To prove (B), assume first that f has a global tract. Then it follows from the preceding that the curves $J(\lambda_n)$ exist, tending to $\{ |z| = 1 \}$ on which $f \to \infty$. Every Γ must intersect infinitely many of the $J(\lambda_n)$ and hence f is unbounded on Γ. Now assume that f is unbounded on every Γ. If C were a non-compact level curve it would contain a Γ on which f would be bounded; hence all the level curves are compact and it follows from (A) that f has a global tract. This completes the proof of Theorem 6.

THEOREM 7. *Let $f \in \mathcal{A}$ and let $\{ D(\varepsilon), \infty \}$ be an arc-tract of f with end K. Let ζ be any point of K, let $\delta > 0$, and let*

$$U(\delta,\zeta) = \{ |z| < 1 \} \cap \{ |z - \zeta| < \delta \}.$$

Then:

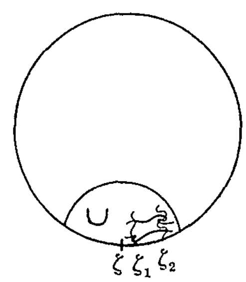
(A) *$f(z)$ assumes every finite value infinitely many times in $U(\delta,\zeta)$.*

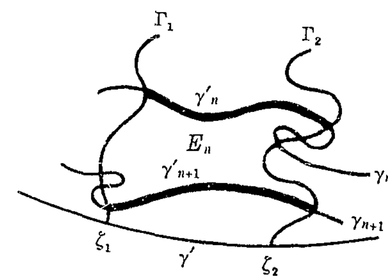
(B) *Let $w = f(z)$ map $\{ |z| < 1 \}$ onto the Riemann surface S, a covering of the w-plane. For any $r > 0$ let the components of S over $\{ |w| < r \}$ be $\Delta(r,1)$, $\Delta(r,2)$, $\cdots$. Let $G(r,n)$ be the domain in $\{ |z| < 1 \}$ corresponding to $\Delta(r,n)$. Then for any given $r > 0$ there are infinitely many of the components $\Delta(r,n)$, say $\Delta(r,n_k)$, $k = 1,2,\cdots$, such that each $\Delta(r,n_k)$ is relatively compact and $G(r,n_k) \subset U(\delta,\zeta)$ for $k \geqslant 1$.*

(C) *Each $D(\varepsilon)$ has infinite connectivity.*

Remark. The relative compactness of $\Delta(r,n)$ (as a subset of S) is equivalent to the relative compactness of $G(r,n)$ as a subset of $\{\,|z|<1\,\}$; that is, the property that the closure of G relative to $\{\,|z|<1\}$ is compact.

Proof. (A) is a simple consequence of (B); so we prove (B).





Let ζ_1 and ζ_2 be two distinct interior points of K such that $|\zeta_i - \zeta| < \delta$, so that arc $\gamma' = (\zeta_1,\zeta_2) \subset K$ lies on the boundary of $U(\delta,\zeta)$. Let Γ_1 and Γ_2 be disjoint Jordan arcs in U, Γ_i ending at ζ_i on which $f \to \infty$ (Theorem 3). Let $\{\gamma_n\}_1^\infty$ be a sequence of disjoint Jordan arcs such that $\gamma_n \subset D(1/n)$ and $\gamma_n \to K$. For n sufficiently large γ_n contains an arc $\gamma'_n \subset U$ which joins a point of Γ_1 to a point of Γ_2 and has no points except its end points in common with Γ_1 and Γ_2. By shortening Γ_i and picking a subsequence of the γ_n' we may assume that:

1) $\Gamma_1 \cup \gamma'_1 \cup \Gamma_2$ is a Jordan arc, a cross-cut of $\{\,|z|<1\,\}$ joining ζ_1 and ζ_2, which, together with γ', bounds a domain H; and 2) that the cross-cuts γ'_n progress monotonically toward γ'; i.e., that γ'_{n+1} separates γ'_n from γ' in H. Let E_n denote the Jordan domain bounded by γ'_n, γ'_{n+1}, and arcs on Γ_1 and Γ_2. Given $r>0$, then for n sufficiently large $E_n \subset U$ and $|f(z)| > r$ on the boundary of E_n. Suppose $|f(z)| \geqslant r$ in all but a finite number of E_n. Then let γ^* be an arc interior to γ'. There would exist a neighborhood V of γ^* such that $|f| \geqslant r$ in V. Then $1/f$ would be holomorphic and bounded in V and tend to zero on a sequence of arcs $\gamma^*_n \subset \gamma'_n$, $\gamma^*_n \to \gamma^*$. Then we would have $1/f \equiv 0$ by Koebe's lemma (see section 1), which is impossible. Thus for infinitely many n: $|f(z)| > r$ on the boundary of E_n and $|f(z)| < r$ at some point of E_n. Each such E_n must contain at least one $G(r,m)$ which will be relatively compact since E_n is relatively compact. Finally (C) follows since the $G(1/\varepsilon,n)$ put an infinity of holes in $D(\varepsilon)$. This completes the proof of Theorem 7.

The result of Theorem 7(B) may be turned into a necessary and sufficient condition for a global tract as follows.

THEOREM 8. *Let $f \in \mathcal{A}$. Then f has a global tract if and only if for every $r > 0$ all $\Delta(r,n)$ are relatively compact.*

Proof. Let f have a global tract. Then by Theorem 6(A) all the level curves of f are compact. Let $r > 0$ be given and choose $r_1 > r$ such that the level set $L(r_1)$ has no multiple points. Each $G(r_1,n)$ is the interior of a Jordan curve which is a component of $L(r_1)$. Hence each $G(r_1,n)$ is relatively compact. Any $G(r,m)$ is contained in some $G(r_1,n)$ and is hence relatively compact.

Conversely, let all $G(r,n)$ be relatively compact. Any level curve $C(\lambda)$ is contained in one $G(r,n)$ if $r > \lambda$ and is hence compact. Also f is unbounded, for if $|f| < M$ then there is just one $G(M,n)$, namely $\{|z| < 1\}$, which is not relatively compact. Then f has a global tract by Theorem 6(A).

THEOREM 9. *Let $f \in \mathcal{A}$ and let $\{\gamma_n\}$ be a sequence of simple arcs in $\{|z| < 1\}$ which tend to an arc $\gamma \subset \{|z| = 1\}$. Let*

$$\mu_n = \max_{z \in \gamma_n} |f(z)|. \tag{4.2}$$

Then

$$\mu = \liminf_{n \to \infty} \mu_n > 0. \tag{4.3}$$

If

$$\mu < \infty \tag{4.4}$$

then at any interior point ζ of γ

$$\limsup_{z \to \zeta} |f(z)| \leqslant \mu. \tag{4.5}$$

Proof. We prove (4.5) first. Let ζ_1 and ζ_2 be two distinct interior points of γ at which f has asymptotic values a_1 and a_2 along curves Γ_1 and Γ_2. We may assume that Γ_1 and Γ_2 start at $z = 0$ and have no other point in common. Let D be the domain bounded by Γ_1, Γ_2, and $\gamma' = (\zeta_1, \zeta_2) \subset \gamma$. Now Γ_1 and Γ_2 must intersect all but a finite number of the γ_n, and we might as well assume it is all. Let $\gamma'_n \subset \gamma_n$ be a cross-cut of D, joining a point of Γ_1 to a point of Γ_2. We can assume, by using a subsequence if necessary, that $\mu_n \to \mu < \infty$ and that γ'_{n+1} separates γ'_n from γ' in D. Let D_n be the subdomain of D bounded

by γ'_n, γ'_{n+1} and two arcs on Γ_1 and Γ_2. Since $f \to a_1, a_2$ on Γ_1, Γ_2 which intersect all γ_n, it follows that $|a_i| \leqslant \mu$. Hence $|f(z)| < \mu + \varepsilon$ on the boundary of D_n for $n > n_0(\varepsilon)$, and (4.5) follows.

If now $\mu = 0$, then it follows from (4.5) and Koebe's lemma (see section 1) that $f \equiv 0$. But $\mathcal{A}$ contains no constant functions, and hence $\mu = 0$ is impossible.

The conclusion (4.3) generalizes Koebe's lemma by replacing the hypothesis f bounded by $f \in \mathcal{A}$.

5. Measurability of $A(S)$. In Theorem 11 in section 6 we shall need the measurability of the set A defined in (3.2). In preparation for this we prove Theorem 10.

THEOREM 10. *Let $f \in \mathcal{A}$ and let S be a Borel set on the sphere. Then $A(S)$ is measurable.*

Remark. Measurable here means Lebesgue measurable as a set in $[0, 2\pi]$. It would of course be quite startling if a holomorphic function could lead to a non-measurable set. Nevertheless, it doesn't seem possible to give a very simple proof of Theorem 10.

Proof. The following relations are obvious, for any finite or countable collection of sets S_n on the sphere.

$$A(\cup S_n) = \cup A(S_n). \tag{5.1}$$

$$A(\cap S_n) \subset \cap A(S_n). \tag{5.2}$$

The reason for inclusion only in (5.2) is that f may have more than one asymptotic value at a given point. For example, $w = i(1+z)/(1-z)$ maps $\{|z| < 1\}$ onto $\mathcal{J}w > 0$ and the function $f(z) = e^{w}$ has two asymptotic values, 0 and ∞ at $z = 1$. Set $S_1 = \{0\}$, $S_2 = \{\infty\}$; then $A(S_1 \cap S_2) = \emptyset$ but $A(S_1) \cap A(S_2) = \{1\}$. This difficulty can be obviated by means of the following result [2, 7].

THEOREM OF BAGEMIHL AND HEINS. *Let $f(z)$ be holomorphic in $\{|z| < 1\}$. The set of points $E \subset \{|z| = 1\}$ at which f has more than one asymptotic value is at most countable.*

It follows then that (5.2) may be improved to

$$\cap A(S_n) = A(\cap S_n) \cup E_1, \qquad E_1 \text{ countable.} \tag{5.3}$$

Then from (5.1) and (5.3) it follows that it suffices to prove the

measurability of $A(S)$ for the two cases: S closed and bounded, $S = \{\infty\}$. For then, if S_0 is the whole sphere, $A = A(S_0)$ is measurable. If $A(S)$ is measurable and $S' = S_o - S$, then

$$A(S') = \{A - A(S)\} \cup \{\textit{countable set}\}$$

is measurable, and so on.

Let now S be closed and bounded. For each $n \geqslant 1$ let $\Delta(n,1), \cdots, \Delta(n, \nu_n)$ be a finite set of open discs of radius 4^{-n} which cover S and is not redundant in the sense that

$$\Delta(n,k) \cap S \neq \emptyset, \qquad 1 \leqslant k \leqslant \nu_n. \tag{5.4}$$

Let $\Delta^*(n,k)$ be the disc of radius $2 \cdot 4^{-n}$ with the same center as $\Delta(n,k)$. The discs Δ and Δ^* may be chosen so that their circumferences contain no projections of the branch-points of $\mathcal{S}$. Here $\mathcal{S}$ is the Riemann surface onto which $w = f(z)$ maps $\{|z| < 1\}$. Then each component of $\mathcal{S}$ over $\Delta(n,k)$ will correspond to a domain $D(n,k,p)$ in $\{|z| < 1\}$ which is bounded by level curves, without multiple points, of $f(z) - a(n,k)$, where $w = a(n,k)$ is the center of $\Delta(n,k)$, and possibly some points on $\{|z| = 1\}$. Now $f \in \mathcal{A}$, hence $f - a \in \mathcal{A} = \mathcal{L}$. Each D is either the interior of a Jordan curve or is bounded by one or more cross-cuts of $\{|z| < 1\}$ and a set $E(n,k,p)$ on $\{|z| = 1\}$. We consider only the second type of D; for given n,k, let these be $D(n,k,p)$, $p \in P(n,k)$. The set $P(n,k)$ may be void, finite, or countably infinite. Now each $E(n,k,p)$ is closed and is hence measurable. $E(n,k,p)$ will contain a finite or countable set $F(n,k,p)$ which consists of the end points of the level curves on the boundary of $D(n,k,p)$. The set

$$H(n,k,p) = E(n,k,p) - F(n,k,p)$$

is then measurable.

Set

$$E(n) = \bigcup_{k,p} E(n,k,p), \quad H(n) = \bigcup_{k,p} H(n,k,p) \tag{5.5}$$

and

$$E = \bigcap_{n=1}^{\infty} E(n), \qquad H = \bigcap_{n=1}^{\infty} H(n). \tag{5.6}$$

These sets are all measurable. Also

$$E(n) = H(n) \cup \{countable\ set\}, E = H \cup \{countable\ set\}. \tag{5.7}$$

Now we prove that

$$H \subset A(S) \subset E, \tag{5.8}$$

from which it follows that $A(S)$ is measurable. If $\zeta \in A(S)$ then $f \to a$ as $z \to \zeta$ along a curve Γ ending at ζ, and also $a \in S$. For each n, $a \in \Delta(n,k)$ for some k and hence a tag end of Γ lies in some $D(n,k,p)$. Then ζ is on the boundary of $D(n,k,p)$; that is, $\zeta \in E(n,k,p)$ for some pair k,p. Hence $\zeta \in E(n)$ for each n, $\zeta \in E$, and thus $A(S) \subset E$.

Now assume $\zeta \in H$. Then for each n there is a pair (k_n,p_n) such that $\zeta \in H(n,k_n,p_n)$. The corresponding domains $D(n,k_n,p_n)$ are such that any two of them must intersect; for if two D's are disjoint then the only boundary points on $\{ |z| = 1 \}$ which they have in common will be end points of boundary cross-cuts, and such points were removed from $E(n,k,p)$ to form $H(n,k,p)$. Since

$$D(m,k_m,p_m) \cap D(n,k_n,p_n) \neq \emptyset \qquad (m,n \geqslant 1) \tag{5.9}$$

it follows that

$$\Delta(m,k_m) \cap \Delta(n,k_n) \neq \emptyset \qquad (m,n \geqslant 1). \tag{5.10}$$

Since the radii of these discs tend to zero, these discs $\Delta(n,k_n)$ tend to a unique point a. Since the covering $\Delta(n,k)$ of S was non-redundant and S is closed, it follows that $a \in S$. It follows from (5.10) and the choice of the radii of the Δ and Δ^* that

$$\Delta^*(n+1,k_{n+1}) \subset \Delta^*(n,k_n). \tag{5.11}$$

One component, S_n of S over $\Delta^*(n,k_n)$ will contain the component over $\Delta(n,k_n)$ which corresponds to $D(n,k_n,p_n)$. Let S_n correspond to the domain D^*_n in $\{ |z| < 1 \}$. Then

$$D(n,k_n,p_n) \subset D^*_n \tag{5.12}$$

and because of (5.9) and (5.11)

$$D^*_{n+1} \subset D^*_n. \tag{5.13}$$

From (5.12) it follows that ζ is a boundary point of each D^*_n. Now let $\Gamma : z = \phi(t)$, $0 \leqslant t < 1$, be a continuous curve such that $\phi(t) \in D^*_n$ for $\tau(n) \leqslant t < 1$. The domains D^*_n are such that $D^*_n \cap \{ \, |z| \leqslant r \, \} = \emptyset$ for $r < 1, n > n_0(r)$, as otherwise it is clear that we would have $f \equiv a$, which is impossible. Hence Γ tends to $\{ \, |z| = 1 \, \}$ and f has the asymptotic value a on Γ. We must show that Γ tends to ζ. From Theorem 4, Γ must tend to some point ζ_1. But ζ is on the boundary of each D^*_n and we can find a curve Γ_1 running through the D^*_n which comes arbitrarily close to both ζ and ζ_1. If $\zeta \neq \zeta_1$, then we would have an arc-tract, which is impossible by Theorem 4. Hence $\zeta \in A(S)$, which completes the proof of (5.8)

Finally we must prove that A_∞ is measurable. A simplified version of the above argument works. We start with the discs $\Delta_n = \{ \, |w| > n \, \}$ and don't have to fool with Δ^*'s. At the very end of the proof we appeal to Theorem 3 rather than Theorem 4.

6. Asymptotic Values on Sets of Positive Measure. We are now in a position to improve on Theorem 2. It will be recalled that A^* denotes the set of points at which f has finite asymptotic values and A_∞ the set where f has the asymptotic value ∞.

THEOREM 11. *Let $f \in \mathcal{A}$. Let γ be an arc of $\{ \, |z| = 1 \, \}$ such that $\gamma \cap A_\infty = \emptyset$. Then* meas $(A^* \cap \gamma) > 0$.

Remarks. By Theorem 10, A^* and hence $A^* \cap \gamma$ are measurable. The inequality meas $(A^* \cap \gamma) <$ meas (γ) is possible, as will be shown by Example 11 in section 10.

Proof. If f is bounded in a neighborhood of some point of γ, then Theorem 11 is an immediate consequence of Fatou's theorem. So we assume that

$$\limsup_{z \to \zeta} |f(z)| = \infty \qquad (\text{all } \zeta \in \gamma). \tag{6.1}$$

That there are functions in $\mathcal{A}$ which satisfy (6.1) at every point of

$\{ \mid z \mid = 1 \}$ but for which $A_\infty = \emptyset$ will be seen from Example 10 in section 10.

Now let ζ_1 and ζ_2 be two distinct points of $A \cap \gamma$. Then f has finite asymptotic values at ζ_1 and ζ_2 and there exists a cross-cut γ_1 of $\{ \mid z \mid < 1 \}$, joining ζ_1 and ζ_2, on which $\mid f \mid \leqslant M$. The arc γ_1 and the arc $\gamma' = (\zeta_1, \zeta_2) \subset \gamma$ bound a domain H in which f is unbounded by (6.1). If $n > M$, H will contain at least one complete component of $\{ z \mid \mid f(z) \mid > n \}$; let these components in H be $D_{n,1}, D_{n,2}, \cdots$. Now suppose f were unbounded in every single set $D_{n,k}$ for all $n > M$ and all k involved. Then $D_{n,1}$ would contain at least one $D_{n+1,i}$, say $D_{n+1,1}$, which in turn would contain $D_{n+2,1}$, and so on. The domains $D_{n,1} \supset D_{n+1,1} \supset \cdots$ determine an asymptotic tract of f with asymptotic value infinity. The end K of this tract is part of γ and by Theorem 3, $K \subset A_\infty$, which is incompatible with our hypothesis $A_\infty \cap \gamma = \emptyset$. Hence there exists some $D_{n,k}$ on which f is bounded; let us denote this domain simply by D_0. It is not necessary that n be an integer, and we may assume that n is such that the level set $L(n)$ contains no multiple points. Then D_0 is bounded by various Jordan curves and cross-cuts, Γ_0, on which $\mid f \mid = n$, and by a set $F \subset \gamma$. Also

$$n < \mid f(z) \mid < N \qquad (z \in D_0). \tag{6.2}$$

The set F is non-void, for otherwise we would have $\limsup \mid f(z) \mid \leqslant n$ at *every* boundary point of D_o implying $\mid f \mid \leqslant n$ in D_o, which won't fit with (6.2). Now the effect of each Jordan curve in Γ_0 is to punch a hole in D_0, which makes the connectivity of D_0 nasty. Add all such holes to D_0 to obtain a domain $D \subset H$ with the properties: D is simply-connected and is bounded by cross-cuts Γ, on which $\mid f \mid = n$, and by F; also

$$\mid f(z) \mid < N \qquad (z \in D). \tag{6.3}$$

Now if Γ contains infinitely many cross-cuts, the diameters of these must approach zero, since $f \in \mathfrak{L} = \mathcal{A}$. It follows readily that the total boundary of D is a Jordan curve. The set F contains no arcs because of (6.1) and (6.3); but it must contain an infinity of points, as otherwise the extended maximum principle would result in $\mid f \mid \leqslant n$, which would contradict (6.2). Actually we shall prove that $F \cap A^* \equiv E_0$ has positive measure.

It is convenient to assume (use a linear transformation on the unit disc if necessary) that $z = 0 \in D$. Let $z = g(Z)$ map $\{ |Z| < 1 \}$ onto D with $g(0) = 0$ and set

$$\phi(Z) = f(g(Z)) = f(z).$$

Then $\phi(Z)$ is bounded in $|Z| < 1$ and by Fatou's theorem has radial limits $\phi(e^{i\theta})$ almost everywhere. Now $\phi(Z)$ is harmonic and may be expressed by the Poisson integral with boundary function $\phi(e^{i\theta})$. Since $|\phi| > n$ in part of $\{ |Z| < 1 \}$ (corresponding to D_0) it follows that $|\phi(e^{i\theta})| > n$ on a set E^* of positive measure. Since D is a Jordan domain, each radial limit $\phi(e^{i\theta})$ corresponds to a point asymptotic value of f in D. The set E^* maps onto $E \subset F$ since $|f| = n$ on Γ.

The proof will be completed by showing that

$$m_e(E) \geqslant m(E^*). \tag{6.4}$$

Here m means measure, m_e exterior measure. That is enough, for F is closed; hence $E_0 = F \cap A^*$ is measurable by Theorem 10 and $E_o \supset E$, and therefore $m(E_o) \geqslant m_e(E)$.

Let $E^*_1 \subset E^*$ be closed and satisfy

$$m(E^*_1) > m(E^*) - \varepsilon. \tag{6.5}$$

Let $\Omega(Z)$ be the harmonic measure in $\{ |Z| < 1 \}$ of the set E^*_1. That is, $1 - \Omega(Z)$ is the harmonic measure of the open set $E^{*\prime}_1$, the sum of the harmonic measures of the intervals comprising $E^{*\prime}_1$. Here the complement (prime) is relative to $\{ |Z| = 1 \}$. Then $2\pi[1 - \Omega(0)] = m(E^{*\prime}_1)$ or

$$2\pi\Omega(0) = m(E^*_1). \tag{6.6}$$

Transfer the harmonic measure $\Omega(Z)$ to D and call it $\omega(z)$. From the normalization of the map g and (6.6) we get

$$2\pi\omega(0) = m(E^*_1). \tag{6.7}$$

Let G be an open set on $\{ |z| = 1 \}$ covering E such that

$$m(G) < m_e(E) + \varepsilon. \tag{6.8}$$

Let $\omega_1(z)$ be the harmonic measure in $\{ |z| < 1 \}$ of G. Then

$$(6.9) \qquad 2\pi\omega_1(0) = m(G).$$

Now the function of $u(z) = \omega_1(z) - \omega(z)$ is harmonic in D. If ζ is an interior point of one of the cross-cuts bounding D, then $\omega_1(\zeta) > 0$ and $\omega(\zeta) = 0$ and hence $u(\zeta) > 0$. If $\zeta \in F \cap G$, then $\omega_1(\zeta) = 1$ and $\limsup_{z\to\zeta}\omega(z) \leqslant 1$, and hence

$$\liminf_{z \to \zeta,\, z \in D} u(z) \geqslant 0.$$

If $\zeta \in F \cap G'$, then ζ corresponds to a point $\xi \in \{ |Z| = 1 \}$ and $\zeta \in G' \Rightarrow \zeta \notin E \Rightarrow \xi \notin E^* \Rightarrow \xi \notin E^*{}_1$, or $\xi \in E^{*\prime}{}_1$. Hence ($E^{*\prime}{}_1$ is open) $\Omega(Z) \to 0$ as $Z \to \xi$; that is, $\omega(z) \to 0$ as $z \to \zeta$, $z \in D$. Thus $\liminf u(z) \geqslant 0$ for $z \to \zeta$, $z \in D$. Thus at *all* boundary points of D, $\liminf u(z) \geqslant 0$ and hence $u(z) \geqslant 0$ in D. In particular $u(0) \geqslant 0$, or

$$(6.10) \qquad \omega_1(0) \geqslant \omega(0).$$

Combining (6.5), (6.7), (6.8), (6.9), and (6.10) we get

$$m_c(E) > m(E^*) - 2\,\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (6.4) follows and the proof of Theorem 11 is complete.

COROLLARY. *Let f and γ satisfy the hypotheses of Theorem 11. Let V be the set of asymptotic values which occur on γ. Then V contains a closed set V_1 of positive harmonic measure.*

This is immediate by applying Priwalow's theorem [20, p. 210] to the function $\phi(Z)$ and its angular limits on the set E^*.

7. A Sufficient Condition for $f \in \mathcal{A}$. The purpose of this section is to prove that if the growth of f is suitably restricted, then $f \in \mathcal{A}$. The precise statement will be found in Theorem 14 below. Theorem 14 depends on Theorems 12 and 13, which we prove first.

We recall the well-known fact that if $u(z)$ is a real-valued uniform harmonic function in $\{ 0 < | z - a | < R \}$ and $u(z) \leqslant M$ then

$$u(z) = c \log | z - a | + U(z)$$

where $c \geqslant 0$ and $U(z)$ is harmonic in $\{ \mid z-a \mid < R \}$. If $c > 0$, we shall say that u has a *negative logarithmic pole* at a.

THEOREM 12. *Let α be fixed, $0 < \alpha \leqslant \pi$, and let*

$$S = \{ z \mid 0 < |z| < 1, |\arg z| < \alpha \}. \tag{7.1}$$

Let $u(z)$ be uniform, real-valued, and harmonic except for possible isolated negative logarithmic poles in S and on the radii $[0, e^{-i\alpha})$ and $[0, e^{i\alpha})$. Let

$$u(z) \leqslant M \qquad (z \in \gamma_n, n \geqslant 1) \tag{7.2}$$

where M is a constant and $\{ \gamma_n \}_1^\infty$ is a sequence of disjoint cross-cuts of S of the following sort:

$$\begin{cases} \gamma_n \text{ joins a point } a_n e^{-i\alpha} \text{ to a point } b_n e^{i\alpha}; \\ \text{here } 0 < a_n, b_n < 1. \end{cases} \tag{7.3a}$$

$$\begin{cases} \gamma_{n+1} \text{ separates } \gamma_n \text{ from the circular arc } \gamma \text{ on the} \\ \text{boundary of } S. \end{cases} \tag{7.3b}$$

$$\min_{z \in \gamma_n} |z| = \rho_n \uparrow 1. \tag{7.3c}$$

Let

$$u(re^{\pm i\alpha}) \leqslant p(r) \qquad (0 \leqslant r < 1) \tag{7.4}$$

where $p(r) \geqslant 0$ is continuous on $[0,1)$. Let $p(r)$ satisfy

$$\int_{a_n}^{a_{n+1}} (1-t)p(t)\,dt + \int_{b_n}^{b_{n+1}} (1-t)p(t)\,dt \leqslant \frac{C}{1-\rho_n} \qquad (n \geqslant 1) \tag{7.5}$$

where C is a constant. Then for each δ, $0 < \delta < \alpha$,

$$\begin{cases} u(z) \text{ is bounded above in the sector} \\ S_\delta = \{ z \mid 0 \leqslant |z| < 1, |\arg z| \leqslant \alpha - \delta \}. \end{cases} \tag{7.6}$$

If $p(r)$ satisfies the stronger condition

$$(7.7)\qquad \int_{a_n}^{a_{n+1}} (1-t)\,p(t)\;dt + \int_{b_n}^{b_{n+1}} (1-t)\,p(t)\;dt \leqslant C \;\; (n \geqslant 1),$$

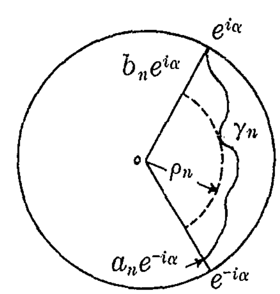
then (7.6) *may be replaced by the stronger conclusion*

$$(7.8)\qquad \lim_{|z| \to 1, z \in S_\delta} \sup \; u(z) \leqslant M.$$

If

$$(7.9)\qquad \int_0^1 (1-t)\,p(t)\;dt < \infty,$$

then (7.7), *and hence* (7.8) *hold.*



Remark. The condition (7.9) is of course more stringent than (7.5) or (7.7); but (7.9) has the great advantage that it doesn't involve $\{ a_n \}$ and $\{ b_n \}$.

The proof of Theorem 12 depends on an appropriate estimate of certain harmonic functions in the sector S. We start with the case $\alpha = \pi/2$.

LEMMA 2. *Let $0 < b_1 < b_2 < 1$; let $p(t) \geqslant 0$ be continuous on $[b_1, b_2]$, and let $U(z)$ be the bounded harmonic function in the semi-disc $D = \{ |z| < 1,\, x = \mathcal{R}\, z > 0 \}$ with boundary values* 0 *except on the interval (ib_1, ib_2), where $U(iy) = p(y)$. Then*

$$(7.10)\qquad U(z) \leqslant \frac{16(1-|z|)}{\pi x^3} \int_{b_1}^{b_2} (1-t)\,p(t)\;dt \qquad (z \in D).$$

Proof. By reflection, $U(z)$ is bounded and harmonic in the half-plane $x > 0$ with boundary values 0 except for $U(iy) = p(y)$ on (ib_1, ib_2) and $U(iy) = -p(1/y)$ on $(i/b_2, i/b_1)$. Then by the Poisson integral for the half-plane,

$$U(z) = \frac{x}{\pi}\int_{b_1}^{b_2} \frac{p(t)\,dt}{x^2+(y-t)^2} - \frac{x}{\pi}\int_{1/b_2}^{1/b_1} \frac{p(1/t)\,dt}{x^2+(y-t)^2}$$

$$= \frac{x}{\pi}\int_{b_1}^{b_2} \frac{p(t)\,dt}{x^2+(y-t)^2} - \frac{x}{\pi}\int_{b_1}^{b_2} \frac{p(t)\,dt}{x^2t^2+(yt-1)^2}$$

$$= \frac{x}{\pi}(1-|z|^2)\int_{b_1}^{b_2} \frac{(1-t^2)\,p(t)\,dt}{[x^2+(y-t)^2]\,[x^2t^2+(yt-1)^2]}.$$

Now $x^2 + (y-t)^2 \geqslant x^2$. If $t \geqslant \frac{1}{2}$ then $x^2t^2 + (yt-1)^2 \geqslant x^2/4$. If $0 \leqslant t \leqslant \frac{1}{2}$ and $|y| < 1$, then $x^2t^2 + (yt-1)^2 \geqslant (yt-1)^2 \geqslant \frac{1}{4}$. Thus for all t in question and for $|z| < 1$, $x^2t^2 + (yt-1)^2 \geqslant x^2/4$. Using also $1-|z|^2 < 2(1-|z|)$ and $1-t^2 < 2(1-t)$, we obtain (7.10).

LEMMA 3. *Let $\frac{1}{2} \leqslant b_1 < b_2 < 1$, $0 < \alpha \leqslant \pi$, let $p(t) \geqslant 0$ be continuous on $[b_1,b_2]$, and let $U(z)$ be the bounded harmonic function in*

$$S = \{ z \mid 0 < |z| < 1, |\arg z| < \alpha \} \tag{7.1}$$

with boundary values zero except that $U(te^{i\alpha}) = p(t)$ on the interval $(b_1e^{i\alpha}, b_2e^{i\alpha})$. Then

$$U(z) \leqslant B_\alpha\, \delta^{-3}\, (1-|z|) \int_{b_1}^{b_2} (1-t)\,p(t)\,dt \qquad (\tfrac{1}{2} \leqslant |z| < 1, |\arg z| \leqslant \alpha - \delta), \tag{7.11}$$

where

$$B_\alpha = \pi^2\, 2^{(5/2)+(3\pi/(2\alpha))}.$$

Here δ is arbitrary as long as $0 < \delta < \alpha$.

Proof. The mapping function $z^{\pi/(2\alpha)}$ takes S onto the semi-disc D of Lemma 2, and from (7.10) we obtain

$$U(z) \leqslant \frac{16(1-r^{\beta})}{\pi r^{3\beta}\cos^3(\beta\theta)} \int_{b_1^{\beta}}^{b_2^{\beta}} (1-t)\,p(t^{1/\beta})\,dt$$

where $z = re^{i\theta}$, $|\theta| < \alpha$, and $\beta = \pi/(2\alpha)$. Then by a simple change of variable in the integral we obtain

$$U(z) \leqslant \frac{16\beta\,(1-r^{\beta})}{\pi r^{3\beta}\cos^3(\beta\theta)} \int_{b_1}^{b_2} t^{\beta-1}(1-t^{\beta})\,p(t)\,dt. \tag{7.12}$$

Since $\frac{1}{2} \leqslant b_1 \leqslant t$ and $\beta \geqslant \frac{1}{2}$, we obtain via the mean value theorem

$$1 - t^{\beta} = (1-t)\,\beta\tau^{\beta-1} < 2^{\frac{1}{2}}\beta\,(1-t). \tag{7.13}$$

Since $\frac{1}{2} \leqslant r$, an analogous estimate holds for $1 - r^{\beta}$. Also

$$t^{\beta-1} \leqslant 2^{\frac{1}{2}}, \qquad r^{3\beta} \geqslant 2^{-3\beta}. \tag{7.14}$$

Finally, since $|\theta| \leqslant \alpha - \delta$, $\beta\,|\theta| \leqslant \frac{\pi}{2} - \beta\,\delta$ and

$$\cos\beta\,\theta \geqslant \sin\beta\,\delta = \sin\frac{\pi\delta}{2\alpha} > \frac{\delta}{\alpha}. \tag{7.15}$$

Using (7.13), (7.14) and (7.15) in (7.12), we obtain (7.11).

Proof of Theorem 12. We may assume that

$$M \leqslant p(r) \qquad (0 \leqslant r < 1); \tag{7.16}$$

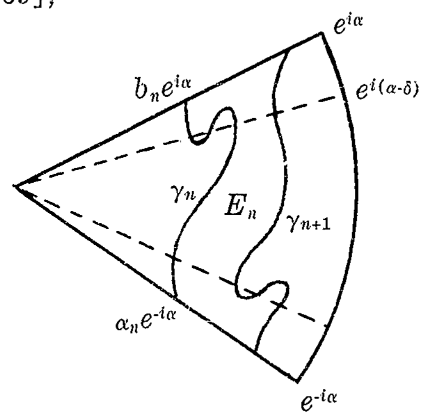
for $p(r)$ may be replaced by $\max(M, p(r))$ without upsetting (7.5), (7.7), or (7.9) except for possibly changing the constant C.

Let E_n denote the subdomain of S which is bounded by γ_n, γ_{n+1}, and the segments $[a_n e^{-i\alpha}, a_{n+1}e^{-i\alpha}]$ and $[b_n e^{i\alpha}, b_{n+1}e^{i\alpha}]$. Then

$$u(z) \leqslant M + v_n(z) \qquad (z \in E_n) \tag{7.17}$$

where $v_n(z)$ is the bounded harmonic function in E_n such that

$v_n(z) = 0$ for $z \in \gamma_n \cup \gamma_{n+1}$ and $v_n\ (re^{\pm i\alpha}) = p(r) - M \geqslant 0$ on the rest of the boundary of E_n. By Carleman's principle of Gebietserweiterung [18, p. 69],



$$v_n(z) \leqslant w_n(z) \qquad (z \in E_n), \tag{7.18}$$

where $w_n(z)$ is the bounded harmonic function in S which vanishes on the boundary of S except that $w_n(re^{\pm i\alpha}) = p(r) - M$ on $[a_n e^{-i\alpha}, a_{n+1}e^{-i\alpha}]$ and $[b_n e^{i\alpha}, b_{n+1}e^{i\alpha}]$. Now $w_n(z)$ is the sum of two harmonic functions of the type $U(z)$ considered in Lemma 3; hence we obtain from (7.11)

$$w_n(z) \leqslant B_\alpha\, \delta^{-3}(1 - |z|)\left\{\int_{a_n}^{a_{n+1}} + \int_{b_n}^{b_{n+1}} (1-t)\,(p(t) - M)\, dt\right\}$$

$$\leqslant B_\alpha\, \delta^{-3}(1 - |z|)\left\{|M| + \int_{a_n}^{a_{n+1}} + \int_{b_n}^{b_{n+1}} (1-t)\,p(t)\, dt\right\},$$

provided $a_n, b_n \geqslant \frac{1}{2}$, $|z| \geqslant \frac{1}{2}$ and $|\arg z| \leqslant \alpha - \delta$. Then from (7.17), (7.18) and (7.3c)

$$u(z) \leqslant M + B_\alpha\, \delta^{-3}(1 - \rho_n)\left\{|M| + \int_{a_n}^{a_{n+1}} + \int_{b_n}^{b_{n+1}} (1-t)\,p(t)\, dt\right\} \tag{7.19}$$

$$(\rho_n \geqslant \tfrac{1}{2},\ z \in \bar{E}_n \cap S_\delta),$$

where $\bar{E}_n$ is the closure of E_n. Now from (7.5) and (7.19)

$$u(z) \leqslant M + C\,B_{\alpha}\,\delta^{-3} + B_{\alpha}\,|\,M\,|\,\delta^{-3}(1-\rho_n) \tag{7.20}$$

$$(\rho_n \geqslant 1/2,\ z \in E_n^{-} \cap S_{\delta})\ .$$

The E_n^{-} for which $\rho_n \geqslant 1/2$ cover that part of S which lies in some annulus $\{\,1-h < |\,z\,| < 1\,\}$. Hence it follows from (7.20) that

$$\limsup_{|\,z\,| \to 1,\ z\, \in\, S_{\delta}} u(z) \leqslant M + C\,B_{\alpha}\,\delta^{-3}, \tag{7.21}$$

and we have proved (7.6). If $p(r)$ satisfies (7.7), then (7.20) is changed to the extent that a factor $(1-\rho_n)$ appears in the second term on the right hand side; then instead of (7.21) we get (7.8). This completes the proof of Theorem 12.

Remark. *The hypothesis that u is harmonic except for possible isolated negative logarithmic poles in S may be replaced by: u is subharmonic in S.* This fact is immediate from the proof of Theorem 12.

THEOREM 13. *Let $f(z)$ be holomorphic in $\{\,|\,z\,| < 1\,\}$. Let S, S_{δ}, and γ_n be as in Theorem 12, and let*

$$\log|\,f(re^{\pm i\alpha})\,| \leqslant p(r) \qquad (0 \leqslant r < 1) \tag{7.22}$$

where $p\ (r) \geqslant 0$ is continuous on $[0, 1)$ and satisfies (7.9).

If

$$|\,f(z)\,| \leqslant C \qquad (z \in \gamma_n, n \geqslant 1) \tag{7.23}$$

where C is a constant, then

$$\limsup_{|\,z\,| \to 1,\ z\, \in\, S_{\delta}} |\,f(z)\,| \leqslant C \qquad (0 < \delta < \alpha). \tag{7.24}$$

If $f(z)$ approaches a finite value a on γ_n, that is, if

$$\lim_{n \to \infty} \left\{ \sup_{z\, \in\, \gamma_n} |\,f(z) - a\,| \right\} = 0, \tag{7.25}$$

then $f(z) \equiv a$ in $\{\,|\,z\,| < 1\,\}$.

Proof. This theorem is an immediate consequence of Theorem 12. To derive (7.24), apply Theorem 12 to $u(z) = \log |f(z)|$ with $M = \log C$; then (7.8) becomes (7.24). Note that if $f \equiv 0$, then Theorem 12 does not apply; but (7.22), (7.23), and (7.24) are all trivial. To derive the last conclusion in Theorem 13, if the conclusion were false (i.e., if $f(z) \not\equiv a$), we could apply Theorem 12 to $u(z) = \log |f(z) - a|$. Then for any constant $B > 0$ it follows from (7.25) that $u(z) < -B$, for $z \in \gamma_n$ and $n > n_0(B)$. With $M = -B$, Theorem 12 yields

$$\limsup_{|z| \to 1, z \in S_\delta} |f(z) - a| \leqslant e^{-B}.$$

Hence $\limsup |f(z) - a| = 0$, and it follows in the familiar fashion from the reflection principle that $f(z) - a \equiv 0$. Here we have used

$$u(re^{\pm i\alpha}) \leqslant \log \{ |f(re^{\pm i\alpha})| + |a| \} \leqslant \log (e^{p(r)} + |a|)$$
$$\leqslant p(r) + \log^+ |a| + \log 2 = p_1(r);$$

and $p_1(r)$ satisfies (7.9) since $p(r)$ does.

It is clear from the last conclusion of Theorem 13 why we referred in section 1 to Theorem 13 as a generalization of Koebe's lemma.

We come now to the sufficient condition for $f \in \mathcal{A}$. *We shall say that $f(z)$, holomorphic in $\{ |z| < 1 \}$, satisfies the condition* (I) *if and only if there exists a set $\Theta \subset [0, 2\pi]$, which is dense in* $[0, 2\pi]$, *such that*

$$\text{(I)} \qquad \int_0^1 (1-r) \log^+ |f(re^{i\theta})| \, dr < \infty \qquad (\theta \in \Theta).$$

No uniformity is implied here; the condition requires only that each individual integral converge. Note that if Θ_0 is a *finite* subset of Θ, then there exists $p(r) \geqslant 0$, continuous on $[0, 1)$, such that

$$\text{(7.26)} \qquad \log^+ |f(re^{i\theta})| \leqslant p(r) \qquad (0 \leqslant r < 1, \theta \in \Theta_0)$$

and such that $p(r)$ satisfies (7.9). Namely,

$$p(r) \equiv \max_{\theta \in \Theta_0} \log^+ |f(re^{i\theta})|$$

will do.

THEOREM 14. *Let $f(z)$ be holomorphic, non-constant, in* $\{ |z| < 1 \}$ *and satisfy* (I). *Then* $f \in \mathcal{A}$.

Proof. The hypothesis that f be non-constant (for any constant satisfies (I)) is necessary since constants are not included in $\mathcal{A}$. We shall prove that $f \in \mathcal{L}$ and then appeal to Theorem 1. Suppose that f satisfies the hypotheses of Theorem 14 and $f \notin \mathcal{L}$. Then there exists a positive λ and a sequence of disjoint compact arcs γ_n in $\{ |z| < 1 \}$, on which $|f| = \lambda$, such that γ_n tends to an arc

$$\gamma = \{ e^{i\theta} \mid \alpha \leqslant \theta \leqslant \beta \}.$$

Let $\alpha < \alpha_1 < \beta_1 < \beta$; $\alpha_1, \beta_1 \in \Theta$. Then it is an immediate consequence of Theorem 13 that f is bounded in each sector S_ε

$$\alpha_1 + \varepsilon \leqslant \arg z \leqslant \beta_1 - \varepsilon$$

of $\{ |z| < 1 \}$. Map S_ε conformally onto $\{ |w| < 1 \}$ and let $f(z) = F(w)$. Now clearly $F(w) \notin \mathcal{L}$, for there will be a sequence of arcs, images of parts of the γ_n, which tend to an arc of $\{ |w| = 1 \}$ and on which $|F| = \lambda$. Also F is non-constant. But this would violate Lemma 1, and hence our assumption $f \notin \mathcal{L}$ must be false. This completes the proof of Theorem 14.

Now the integrals in condition (I) immediately suggest some connection with the Schmiegungsfunktion

$$m(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \qquad (0 \leqslant r < 1) \tag{7.27}$$

of Nevanlinna. Such a relation is easily found. Let

$$\sigma(\theta) = \int_0^1 (1-r) \log^+ |f(re^{i\theta})| \, dr. \tag{7.28}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} \sigma(\theta) \, d\theta = \int_0^1 (1-r) m(r) \, dr \tag{7.29}$$

without restriction on f. Hence if the integral on the right of (7.29) converges, then $\sigma(\theta)$ will be finite for almost all θ. Hence if we set

$$\text{(II)} \qquad \int_0^1 (1-r)\, m(r)\, dr < \infty,$$

then it follows that

$$(7.30) \qquad \text{(II)} \Rightarrow \text{(I)}.$$

We may also introduce a third condition, namely,

$$\text{(III)} \qquad \int_0^1 (1-r) \log^+ M(r)\, dr < \infty,$$

where $M(r)$ is the maximum modulus of f. Since $m(r) \leqslant \log^+ M(r)$, it is clear that

$$(7.31) \qquad \text{(III)} \Rightarrow \text{(II)}.$$

It is also immediately clear that if $f \not\equiv 0$ then (III) is equivalent to

$$(7.32) \qquad \int_0^1 (1-r) \log M(r)\, dr < \infty,$$

for the integral in (7.32) will differ from that in (III) by a finite constant.

The hypothesis (I) *in Theorem 14 may be replaced by either* (II) *or* (III).

It will be shown in Examples 6, 7, 8, 9 in section 10 that neither implication (7.30) or (7.31) can be reversed.

We now take up a condition, related to (III), in terms of the Taylor coefficients of f. If f is bounded in $\{ |z| < 1 \}$ we shall say that the order of f is zero. If f is unbounded, the order ρ is defined by

$$(7.33) \qquad \rho = \limsup_{r \to 1} \frac{\log \log M(r)}{\log \dfrac{1}{1-r}}.$$

Another way of putting it is as follows: if $f \not\equiv 0$, then ρ is the infimum of the real λ such that

$$M(r) = O\{\exp[(1-r)^{-\lambda}]\} \qquad (r \to 1).$$

It should be noted that the order of f as defined by (7.33) is not the same as the order of f as defined by Nevanlinna [17, pp. 138-140]. Nor is it the same as Hadamard's order ω of f on $\{\,|z|=1\,\}$; in general if $\rho > 0$ then $\omega = \infty$. For our purposes the order ρ is pertinent. We use the following theorem.

THEOREM 15. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1). \tag{7.34}$$

Then the order ρ *of* f *is given by either*

$$\frac{\rho}{\rho+1} = \limsup_{n\to\infty} \frac{\log^+ \log^+ |a_n|}{\log n} \tag{7.35}$$

or

$$\rho = \limsup_{n\to\infty} \frac{\log^+ \log^+ |a_n|}{\log n - \log^+ \log^+ |a_n|}. \tag{7.36}$$

Proof. It should be noted that (7.35) or (7.36) does define a ρ in the range $0 \leqslant \rho \leqslant \infty$, for the radius of convergence of (7.34) is at least one and hence

$$\log^+ \log^+ |a_n| < \log n \qquad (n > n_0). \tag{7.37}$$

The equivalence of (7.35) and (7.36) is trivial; $\rho/(\rho+1)$ is an increasing function of ρ for $\rho \geqslant 0$.

Next note that if the sequence a_n is bounded, then f is dominated by a geometric series and it follows immediately from (7.33) that $\rho = 0$. But (7.36) also gives $\rho = 0$. So it is enough to consider the case $\limsup |a_n| = \infty$; in this case $M(r)$ is unbounded. For the significant n in this case all the $\log^+$ in (7.35) and (7.36) may be replaced by log.

Now suppose that

$$\log M(r) \leqslant A(1-r)^{-\lambda} \qquad (\lambda > 0, A \text{ const}). \tag{7.38}$$

Then, from the Cauchy estimate,

$$\log^+ |a_n| \leqslant A(1-r)^{-\lambda} - n \log r. \tag{7.39}$$

Both terms on the right of (7.39) are positive and the $+$ on the log is justified. We choose (for $n > A$)

$$1 - r = \left(\frac{A}{n}\right)^{1/(\lambda+1)}$$

and obtain from (7.39)

$$\log^+ |a_n| \leqslant 2A^{1/(\lambda+1)} n^{\lambda/(\lambda+1)} \{1 + O(n^{-1/(\lambda+1)})\} \qquad (n \to \infty). \tag{7.40}$$

It follows immediately from (7.40) that

$$\limsup \frac{\log^+ \log^+ |a_n|}{\log n} \leqslant \frac{\lambda}{\lambda+1}.$$

If f is of finite order ρ then, see (7.38), we may use any $\lambda > \rho$ and hence

$$\limsup \frac{\log^+ \log^+ |a_n|}{\log n} \leqslant \frac{\rho}{\rho+1}. \tag{7.41}$$

In case $\rho = \infty$ (7.41) is valid because of (7.37), and the proof of (7.35) is complete in the case

$$\limsup \frac{\log^+ \log^+ |a_n|}{\log n} = 1.$$

Now suppose that last is not the case; then there are constants σ such that

$$\log^+ |a_n| \leqslant n^{\mu/(\mu+1)} = n^{\sigma} \quad (0 < \sigma < 1, n \geqslant n_0). \tag{7.42}$$

Then

$$M(r) \leqslant A + \sum_{n_0}^{\infty} e^{n^\sigma} r^n. \tag{7.43}$$

Now it is readily seen that if $g(x) = x^\sigma + x \log r$, then

$$\max_{0 \leqslant x < \infty} g(x) = \{\sigma^{\sigma/(1-\sigma)} - \sigma^{1/(1-\sigma)}\}\left(\log\frac{1}{r}\right)^{-\mu} .$$

Thus

$$\log\,(e^{n^\sigma} r^n) \leqslant B\,(\sigma)\left(\log\frac{1}{r}\right)^{-\mu},$$

and

$$M(r) \leqslant A + N \exp\left\{ B(\sigma)\left(\log\frac{1}{r}\right)^{-\mu}\right\} + \sum_{N+1}^{\infty} e^{n^\sigma} r^n. \tag{7.44}$$

Now choose

$$N = \left[\left(\frac{1}{2}\log\frac{1}{r}\right)^{-1/(1-\sigma)}\right], \tag{7.45}$$

where [ ] denotes the greatest integer not exceeding. Then if $n \geqslant N+1$,

$$n \geqslant \left(2/\log\frac{1}{r}\right)^{1/(1-\sigma)},$$

and that inequality is equivalent to

$$n^\sigma + n \log r \leqslant \frac{n}{2} \log r. \tag{7.46}$$

Then

$$\sum_{N+1}^{\infty} e^{n^{\sigma}} r^{n} \leqslant \sum_{N+1}^{\infty} r^{n/2} = \frac{r^{(N+1)/2}}{1-r^{1/2}} . \tag{7.47}$$

Now

$$\log r^{(N+1)/2} = -\frac{N+1}{2} \log \frac{1}{r} < -\frac{1}{2}\left(2/\log \frac{1}{r}\right)^{1/(1-\sigma)} \log \frac{1}{r}$$

$$= -2^{\mu}\left(\log \frac{1}{r}\right)^{-\mu} . \tag{7.48}$$

Also, as $r \uparrow 1$,

$$1-r = \left(\log \frac{1}{r}\right)\left[1+O\left(\log \frac{1}{r}\right)\right],$$

$$1-\sqrt{r} = \frac{1}{2}\left(\log \frac{1}{r}\right)\left[1+O\left(\log \frac{1}{r}\right)\right], \tag{7.49}$$

$$\log (1-\sqrt{r}) = -\log 2 + \log\log \frac{1}{r} + O\left(\log \frac{1}{r}\right),$$

and hence

$$\log \frac{r^{(N+1)/2}}{1-r^{1/2}} \leqslant -2^{\mu}\left(\log \frac{1}{r}\right)^{-\mu} + \log 2 - \log\log \frac{1}{r} + O\left(\log \frac{1}{r}\right)$$

$$= -2^{\mu}\left(\log \frac{1}{r}\right)^{-\mu}[1+o(1)].$$

From this it follows that the series (7.47) is $o(1)$ as $r \uparrow 1$. Then from (7.44) and (7.45) we get

$$M(r) \leqslant A_1 + \left(2/\log \frac{1}{r}\right)^{1/(1-\sigma)} \exp\left\{B(\sigma)\left(\log \frac{1}{r}\right)^{-\mu}\right\}$$

and it follows that

$$\log M(r) \leqslant B(\sigma)\left(\log \frac{1}{r}\right)^{-\mu}[1+o(1)]$$

and hence, from (7.49)

$$\log M(r) \leqslant B(\sigma)\ (1-r)^{-\mu}\ [1+o(1)].$$

Then

$$\log \log M(r) \leqslant \mu \log \frac{1}{1-r} + O(1),$$

and it follows from (7.33) that $\rho \leqslant \mu$. Recalling (7.42), we may choose $\mu/(\mu+1)$ arbitrarily close to

$$\limsup \frac{\log^+ \log^+ |\, a_n \,|}{\log n},$$

and hence the order ρ of f satisfies

$$\frac{\rho}{\rho+1} \leqslant \limsup \frac{\log^+ \log^+ |\, a_n \,|}{\log n}. \tag{7.50}$$

Combining (7.41) and (7.50) the proof of Theorem 15 is complete.

Now it follows from (7.33) that condition (III) is satisfied if $\rho < 2$. Hence, from Theorem 14, Theorem 15, (7.30), and (7.31) we immediately have the following result.

THEOREM 16. *Let $f(z)$ be non-constant,*

$$f(z) = \sum_{0}^{\infty} a_n z^n \qquad (|\, z \,| < 1) \tag{7.34}$$

and let λ be a constant such that

$$0 < \lambda < \frac{2}{3},\ \log^+ |\, a_n \,| < n^{\lambda} \qquad (n > n_0). \tag{7.51}$$

Then $f \in \mathcal{A}$.

A bizarre way of interpreting this result is as follows. If $\{ a_n \}$ satisfies (7.51), then for a dense set of θ the trigonometric series

$\sum_{0}^{\infty} a_n e^{ni\theta}$ is summable in a "skewgee Abel sense."

8. Other Sufficient Conditions for $f \in \mathcal{A}$.

The next theorem involves the concept of normal function as defined by Lehto and Virtanen [10]: *the function $f(z)$, meromorphic in $\{ |z| < 1 \}$, is normal if and only if the set of functions $f(T(z))$, where T ranges over the linear transformations preserving $\{ |z| < 1 \}$, is a normal family.* For instance, if f omits three values it is normal. We shall need two results of Lehto and Virtanen [10, Theorems 2 and 3].

THEOREM L V_2. *If f is a normal meromorphic function in $\{ |z| < 1 \}$ and has the asymptotic value a along an arc ending at ζ, $|\zeta| = 1$, then f has the angular limit a at ζ.*

THEOREM L V_3. *If f is meromorphic in $\{ |z| < 1 \}$, then it is normal if and only if*

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leqslant \frac{C}{1 - |z|^2} \qquad (|z| < 1) \tag{8.1}$$

where C is a constant.

DEFINITION. *$f(z)$ belongs to the class $\mathcal{N}$ if and only if f is holomorphic, non-constant, in $\{ |z| < 1 \}$, and normal.*

THEOREM 17. *$\mathcal{N} \subset \mathcal{A}$ and the inclusion is proper. Also, if $f \in \mathcal{N}$, then:*

1° *Given ζ, $|\zeta| = 1$, f has at most one asymptotic value at ζ. If f has the asymptotic value a at ζ then f has the angular limit a at ζ.*

2° *f has no arc-tracts.*

Remark. Lehto and Virtanen gave an example [10, p. 58] of a normal function without any asymptotic values; but that function is meromorphic, not holomorphic, and Theorem 17 shows that such must be the case.

Proof. We assume that the non-constant holomorphic function f satisfies (8.1) and prove that $f \in \mathcal{A}$. Consider the spherical characteristic of Shimizu and Ahlfors [18, p. 177]

$$T_S(r) \equiv \int_0^r \frac{A(t)}{t}\, dt \qquad (0 \leqslant r < 1)$$

where

$$A(t) \equiv \frac{1}{\pi} \iint_{r<t} \frac{|f'(re^{i\theta})|^2}{(1+|f(re^{i\theta})|^2)^2}\, r\, dr\, d\theta \quad (0 \leqslant t < 1).$$

Then from (8.1)

$$A(t) \leqslant C^2 \int_0^t \frac{2r\, dr}{(1-r^2)^2} = \frac{C^2 t^2}{1-t^2} \leqslant \frac{C^2 t}{1-t},$$

and hence

$$T_S(r) \leqslant C^2 \int_0^r \frac{dt}{1-t} = C^2 \log \frac{1}{1-r}.$$

Since $f(z)$ is holomorphic,

$$\text{(8.2)} \quad m(r) = T(r) = T_S(r) + O(1) < C^2 \log \frac{1}{1-r} + O(1) \qquad (0 \leqslant r < 1),$$

where $T(r)$ is the original Nevanlinna characteristic of f. It is immediately clear from (8.2) that $m(r)$ satisfies condition (II), and hence $f \in \mathcal{A}$.

Next 1° follows immediately from Theorem LV_2. It is then clear that the inclusion $\mathcal{N} \subset \mathcal{A}$ is proper. For in [15] it was shown that there exist holomorphic functions of arbitrarily slow growth (and hence some that satisfy condition (III)) without any radial limits.

Finally, to prove 2°, if f had an arc-tract with arc K, then the associated asymptotic value would be ∞ by Theorem 4. By 1° and Theorem 3 f would have the angular limit ∞ at every point of K and the meromorphic function $1/f$ would have the angular limit zero at every point of K. By the theorem of Lusin and Priwalow [20, p. 212], this would imply $1/f \equiv 0$, which is impossible. If f is normal because it omits two finite values, then 2° would also follow trivially from Theorem 7.

Another sufficient condition results from a Theorem of Tsuji [21], of which we quote only part, as follows.

THEOREM OF TSUJI. *Let $f(z)$ be meromorphic in $\{|z| < 1\}$ and satisfy*

$$(8.3) \qquad \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\,d\theta \leqslant C \qquad (0 \leqslant r < 1).$$

where C is a finite constant, so that the circle $\{\,|z| = r\,\}$ is mapped on a curve of length $\leqslant C$ on the sphere.

Then for almost all θ the radius $[0, e^{i\theta})$ is mapped onto a rectifiable curve on the sphere, and hence the radial limit of $f(z)$ exists almost everywhere.

The rest of Tsuji's theorem is to the effect that any f satisfying (8.3) comes "close" to having angular limits almost everywhere. Now the proof of Tsuji's theorem comes from the fact that (8.3) implies

$$(8.4) \qquad \int_0^{1} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\,dr < \infty$$

for almost all θ. Certainly if f is holomorphic and non-constant in $\{\,|z| < 1\,\}$ and satisfies (8.3) then it belongs to $\mathcal{A}$. But that condition may be weakened to: (8.4) holds on a dense set of radii. We can even do somewhat better than that, as follows.

THEOREM 18. *Let $f(z)$ be holomorphic and non-constant in $\{\,|z| < 1\,\}$ and let $\mu(t)$ be a real positive increasing function on $[0, \infty)$. If*

$$(8.5) \qquad \int_0^{1} \frac{|f'(re^{i\theta})|}{\mu(|f(re^{i\theta})|)}\,dr < \infty \qquad (\theta \in \Theta),$$

then the radial limit

$$\lim_{r \uparrow 1} f(re^{i\theta}) = f(e^{i\theta})$$

exists for each $\theta \in \Theta$. If Θ is dense in $[0, 2\pi]$, then $f \in \mathcal{A}$.

Remark. This result may be generalized by using a different $\mu(t)$ for each $\theta \in \Theta$, as will be clear from the proof.

Proof. We shall show that for any fixed θ for which (8.5) converges, the corresponding radial limit exists. We may assume that $\theta = 0$. Suppose that $\lim_{x\uparrow 1} f(x)$ does not exist. If $f(x)$ is unbounded on $[0,1)$, then $\limsup |f(x)| = \infty$, $\liminf |f(x)| = \lambda < \infty$. Thus the curve $\Gamma : w = f(x)$, $0 \leqslant x < 1$, will contain infinitely many disjoint arcs Γ_n, each lying in $\{ c \leqslant |w| \leqslant 2c \}$, where $c = \lambda + 1 > 0$, and joining a point of $\{ |w| = c \}$ to a point of $\{ |w| = 2c \}$. Let Γ_n correspond to the interval $[x_n, x_n + h_n]$. Then

$$0 < x_1 < x_1 + h_1 < x_2 < x_2 + h_2 < \cdots \uparrow 1, \tag{8.6}$$

$$|f(x_n + h_n) - f(x_n)| \geqslant c > 0 \qquad (n \geqslant 1) \tag{8.7}$$

and

$$|f(x)| \leqslant C < \infty \quad (x_n \leqslant x \leqslant x_n + h_n, n \geqslant 1). \tag{8.8}$$

In this case $C = 2c$. If $f(x)$ is bounded on $[0,1)$ but $\lim f(x)$ does not exist, then it is easy to see that there exist x_n, h_n, c and C such that (8.6), (8.7) and (8.8) are satisfied. Then

$$\int_{x_n}^{x_n+h_n} \frac{|f'(t)|\, dt}{\mu(|f(t)|)} \geqslant \frac{1}{\mu(C)} \left| \int_{x_n}^{x_n+h_n} f'(t)\, dt \right|$$

$$= \frac{1}{\mu(C)} |f(x_n + h_n) - f(x_n)| \geqslant \frac{c}{\mu(C)},$$

and hence

$$\int_0^1 \frac{|f'(t)|\, dt}{\mu(|f(t)|)} \geqslant \sum \int_{x_n}^{x_n+h_n} = \infty.$$

This completes the proof of Theorem 18.

Another type of criterion is furnished by the following theorem.

THEOREM 19. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n} \qquad (|z| < 1) \tag{8.9}$$

where

$$\liminf_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} > 3. \tag{8.10}$$

Then $f \in \mathcal{A}$. *If in addition*

$$\sum_{n=1}^{\infty} |a_n| = \infty, \tag{8.11}$$

then there exist two sets, Θ_+ *and* Θ_-, *each dense in* $[0,2\pi]$, *such that*

$$\lim_{r \uparrow 1} u(re^{i\theta}) = \begin{cases} +\infty, & (\theta \in \Theta_+) \\ -\infty, & (\theta \in \Theta_-) \end{cases} \tag{8.12}$$

where $u = \mathcal{R}f$. *Similar results hold for* $v = \mathcal{I}f$. *In particular, when* (8.11) *holds,* A_∞ *is dense on* $\{ |z| = 1 \}$.

Proof. If (8.11) fails, then f is bounded and $f \in \mathcal{A}$ by Fatou's theorem. So we assume (8.11) and prove the first relation in (8.12). Let $a_n = A_n e^{i\alpha_n}$; then

$$u(re^{i\theta}) = \sum_{n=1}^{\infty} A_n r^{\lambda_n} \cos(\lambda_n \theta + \alpha_n). \tag{8.13}$$

From (8.10) we may assume (throw out some initial terms in the series (8.9)) that

$$\inf_{n \geqslant 1} \frac{\lambda_{n+1}}{\lambda_n} > 3.$$

Then there exists δ, $0 < \delta < \pi/2$, such that

$$\frac{\pi - 2\delta}{\lambda_n} > 3 \frac{\pi}{\lambda_{n+1}} \qquad (n \geqslant 1). \tag{8.14}$$

Set $\sin \delta = \eta > 0$. Now $\cos(\lambda_n \theta + \alpha_n) > 0$ in various intervals of length π/λ_n. Chopping off δ/λ_n from each end of such an interval, we obtain a closed interval I_n of length $(\pi - 2\delta)/\lambda_n$ in which

$$\cos(\lambda_n \theta + \alpha_n) \geqslant \eta \qquad (\theta \in I_n). \tag{8.15}$$

By (8.14) I_n must contain two consecutive full arches of the graph of $\cos(\lambda_{n+1}\theta + \alpha_{n+1})$, and one of these arches will contain an interval of the type I_{n+1}.

Now let Θ_+ be defined as the set of θ for which the upper result in (8.12) holds. Let γ be a subinterval of $[0, 2\pi]$. Then there is an n_1 such that γ contains an interval I_{n_1}. Using the argument above we get a telescoping sequence of closed intervals,

$$\gamma \supset I_{n_1} \supset I_{n_1+1} \supset \cdots \supset I_n \supset \cdots,$$

for which (8.15) is satisfied for $n \geqslant n_1$. Let θ_0 be the point common to all the I_n. Then by (8.15) and (8.13),

$$u(re^{i\theta_0}) \geqslant - \sum_{1}^{n_1-1} A_n + \eta \sum_{n_1}^{\infty} A_n r^{\lambda_n}.$$

By (8.11) the last sum tends to $+\infty$ as $r \uparrow 1$. Hence $\theta_0 \in \Theta_+$ and, since γ was arbitrary, Θ_+ is dense, which completes the proof of Theorem 19.

Examples 4 and 5 in section 10 are related to Theorem 19.

9. The Number of Asymptotic Values at One Point.

In the proof of Theorem 10 we have already quoted the theorem of Bagemihl and Heins to the effect that the set of points on $\{|z| = 1\}$ at which a holomorphic function has more than one asymptotic value is at most countable. But of course at one point f may have infinitely many asymptotic values. For instance, consider Grosz' [6] entire function $F(\zeta)$, for which *every* complex number is an asymptotic value. Slit the ζ-plane along one path on which $F \to \infty$ and map the slit plane onto $\{|z| < 1\}$ so that $\zeta = \infty$ corresponds to $z = 1$. Set $f(z) = F(\zeta)$. Clearly $f(z) \in \mathcal{A}$, since f is continuous at each point of $\{|z| = 1\}$ except at $z = 1$. But, at $z = 1$, f has every complex number as an asymptotic value. Also observe that f satisfies condition (I).

The object of this section is to show that if f satisfies a condition of the type (III) then there is a limitation on the number of asymptotic tracts which can be associated with one point on $\{|z| = 1\}$. The precise statement is in Theorem 22. The argument depends on

Theorem 20, an adaptation of the type of harmonic dominance argument used by F. and R Nevanlinna [19], and a slight extension, Theorem 21, of Ahlfors' proof of the Denjoy conjecture on the number of asymptotic tracts of entire functions.

THEOREM 20. *Let $u(z)$ be real-valued and harmonic, except for isolated negative logarithmic poles, in $\{ |z| < 1 \}$. Let $p(r)$ be a positive increasing function on* $[0,1)$ *such that*

$$u(z) \leqslant p(r) \qquad (0 \leqslant |z| = r < 1). \tag{9.1}$$

Let C_1 and C_2 be two Jordan arcs in $\{ |z| < 1 \}$ joining $z = 0$ to $z = 1$, which are disjoint except for their common end points. Let D be the Jordan domain bounded by C_1 and C_2. Let

$$u(z) \leqslant M \qquad (z \in C_1 \cup C_2), \tag{9.2}$$

where M is a constant.

Let λ, $0 \leqslant \lambda \leqslant 1$, be a constant such that

$$\int_0^1 (1-t)^{\lambda} p(t)\, dt < \infty. \tag{9.3}$$

Then

$$u(z) \leqslant o(|1-z|^{-\lambda-1}) \qquad (z \in D, z \to 1). \tag{9.4}$$

If $\lambda = 0$ we have the stronger result

$$u(z) \leqslant M \qquad (z \in D). \tag{9.5}$$

Remarks. It will be noted in the course of the proof that it breaks down and yields nothing if $\lambda > 1$. It should be observed that there is no restriction on the curves C_1 and C_2; they may approach $z = 1$ tangentially, for instance. That lack of restriction is essential since we are interested in asymptotic values along *any* curves. *If* C_1 and C_2 lie in some angle $|\arg(1-z)| \leqslant \frac{\pi}{2} - \delta$, $0 < \delta < \frac{\pi}{2}$, then for $|1-z| \leqslant \sin\delta$

$$|z| \leqslant 1 - \tfrac{1}{2}|1-z| \sin\delta,$$

and, using (9.12) and (9.14) (which are trivial) we get

$$u(z) \leqslant p(|z|) \leqslant p(1 - \tfrac{1}{2} |1 - z| \sin \delta) = o(|1 - z|^{\lambda - 1}) \qquad (z \in D).$$

Proof. Map $\{ |z| < 1 \}$ onto $\{ \mathcal{R}\zeta > 0 \}$ by $\zeta = (1 + z)/(1 - z)$ so that $z = 0,1$ correspond to $\zeta = 1, \infty$. Then C_1 and C_2 correspond to arcs Γ_1 and Γ_2 from $\zeta = 1$ to $\zeta = \infty$ and D corresponds to Δ, bounded by Γ_1 and Γ_2. Let $u(z) = U(\zeta)$. Let $\zeta = \rho e^{i\phi}$. Then

$$r^2 = |z|^2 = 1 - \frac{4\rho \cos \phi}{1 + 2\rho \cos \phi + \rho^2} \leqslant 1 - \frac{4\rho \cos \phi}{(1 + \rho)^2},$$

and hence

$$r < 1 - \frac{2\rho \cos \phi}{(1 + \rho)^2}. \tag{9.6}$$

Let now $\rho_0 > 0$ be a constant such that $\{ |\zeta| \leqslant \rho_0 \} \cap \Delta = \emptyset$. Then for $\rho \geqslant \rho_0$

$$\frac{\rho^2}{(1 + \rho)^2} \geqslant \frac{\rho_0^2}{(1 + \rho_0)^2} = \frac{c}{2},$$

and hence

$$\frac{\rho}{(1 + \rho)^2} \geqslant \frac{c}{2\rho} \qquad (\rho \geqslant \rho_0). \tag{9.7}$$

We note for future reference that since $\zeta = 1$ is a boundary point of Δ then $\rho_0 \leqslant 1$, and hence

$$0 < c \leqslant \tfrac{1}{2}; \qquad \frac{c}{\rho} \leqslant \frac{c}{\rho_0} \leqslant \tfrac{1}{2} \qquad (\rho \geqslant \rho_0). \tag{9.8}$$

Combining (9.6) and (9.7),

$$r < 1 - \frac{c \cos \phi}{\rho} \qquad (\rho \geqslant \rho_0),$$

and hence from (9.1)

$$U(\zeta) \leqslant p\left(1 - \frac{c \cos \phi}{\rho}\right) \equiv P(\rho, \phi) \qquad (\rho \geqslant \rho_0). \tag{9.9}$$

Now let $R > 1$; let H_R denote the half-disc $\{ \mid \zeta \mid < R, \mathcal{R}\, \zeta > 0 \}$, and let Δ^*_R be *any* component of $H_R \cap \Delta$. Then Δ^*_R will be bounded by one or more cross-cuts of H_R lying on $\Gamma_1 \cup \Gamma_2$ and various arcs of $\{ \mid \zeta \mid = R \}$. There is one domain Δ_R bounded by a single one of those cross-cuts and a single arc $\gamma \subset \{ \mid \zeta \mid = R, \mathcal{R}\, \zeta > 0 \}$ such that $\Delta_R \supset \Delta^*_R$. By (9.9), and since $U(\zeta) \leqslant M$ on $\Gamma_1 \cup \Gamma_2$, it follows from the maximum principle that

$$U(\zeta) \leqslant M + U_1(\zeta) \qquad (\zeta \in \Delta_R) \tag{9.10}$$

where $U_1(\zeta)$ is the bounded harmonic function in H_R such that $U_1(\zeta) = P(\rho,\phi) + \mid M \mid$ on γ and $U_1 = 0$ on the remainder of the boundary of H_R. By the reflection principle, U_1 is harmonic in $\{ \mid \zeta \mid < R \}$ and, combining the Poisson integrals over γ and its reflection, it follows that

$$U_1(\rho e^{i\phi}) = \frac{2}{\pi} R\rho (R^2\text{-}\rho^2) \cos \phi$$
$$\times \int_\gamma \frac{\{P(R, t) + \mid M \mid \} \cos t \, dt}{[R^2\text{-}2R\, \rho \cos (\phi\text{-}t) + \rho^2]\ [R^2 + 2R\, \rho \cos (\phi + t) + \rho^2]}.$$

Then in H_R, where $\cos \phi > 0$, it follows that

$$U_1(\rho e^{i\phi}) \leqslant \frac{2R\rho\ (R + \rho)}{\pi (R - \rho)^3} \int_\gamma \{ P(R, t) + \mid M \mid \} \cos t \, dt$$

or

$$U_1(\rho e^{i\phi}) \leqslant \frac{2R\rho\ (R + \rho)}{\pi (R - \rho)^3} \left\{ 2 \mid M \mid + \int_\gamma P(R, t) \cos t \, dt \right\}. \tag{9.11}$$

Now set

$$q(t) = p(1 - ct) \qquad (0 < t \leqslant 1/c). \tag{9.12}$$

Then $q(t)$ is positive and decreasing, and (9.3) is equivalent to

$$\int_0^1 t^\lambda q(t) \, dt < \infty. \tag{9.13}$$

From (9.13)

$$q(r)\int_0^r t^\lambda\,dt \leqslant \int_0^r t^\lambda\,q(t)\,dt = o(1) \qquad (r \downarrow 0)$$

and hence

$$q(t) = o(t^{-\lambda-1}) \qquad (t \downarrow 0). \tag{9.14}$$

We note from (9.12), (9.9), and (9.8) that

$$P(R, t) = q\left(\frac{\cos t}{R}\right),$$

which is certainly defined for $R > 1$. Then

$$\int_\gamma P(R, t)\cos t\,dt = \int_\gamma q\left(\frac{\cos t}{R}\right)\cos t\,dt$$

$$\leqslant \int_0^{\pi/2} q\left(\frac{\cos t}{R}\right)\cos t\,dt = 2\int_0^1 u\,q\left(\frac{u}{R}\right)\frac{du}{\sqrt{1-u^2}},$$

and hence

$$\int_\gamma P(R, t)\cos t\,dt \leqslant 2\int_0^1 u\,q\left(\frac{u}{R}\right)\frac{du}{\sqrt{1-u}}. \tag{9.15}$$

Now this estimate is nothing unless the integral on the right converges; and this convergence follows from (9.13), since $\lambda \leqslant 1$. Then

$$\int_0^1 u\,q\left(\frac{u}{R}\right)\frac{du}{\sqrt{1-u}} \leqslant \sqrt{2}\int_0^{1/2} u\,q\left(\frac{u}{R}\right)du + q\left(\frac{1}{2R}\right)\int_{1/2}^1 \frac{du}{\sqrt{1-u}}$$

$$= \sqrt{2}\,R^2\int_0^{1/(2R)} t\,q(t)\,dt + \sqrt{2}\,q\left(\frac{1}{2R}\right)$$

$$\leqslant \sqrt{2}\, R^2 \left(\frac{1}{2R}\right)^{1-\lambda} \int_0^{1/(2R)} t^\lambda q(t)\, dt + o(R^{\lambda+1})$$

where we have used (9.14). Now the last integral is $o(1)$ as $R \to \infty$, by (9.13), and it follows that

$$\int_0^1 u\, q\left(\frac{u}{R}\right) \frac{du}{\sqrt{1-u}} = o(R^{\lambda+1}) \qquad (R \to \infty). \tag{9.16}$$

Now from (9.10), (9.11), (9.15), and (9.16),

$$U(\rho e^{i\phi}) \leqslant M + \frac{4R\rho(R+\rho)}{\pi(R-\rho)^3}\{\, o(R^{\lambda+1})\,\} \quad (\rho e^{i\phi} \in \Delta,\ \rho < R). \tag{9.17}$$

It should perhaps be pointed out that the same uniform estimate holds in all Δ_R (there may be several Δ_R for a given R) and hence only $\Delta \cap \{\, |\, \zeta\, | < R\,\}$ appears in (9.17); that is because the particular choice of Δ_R, i.e., of γ, disappeared when γ was replaced by $(-\pi/2, \pi/2)$ immediately before (9.15). Also the o term in (9.17) is independent of ρ.

If, for $\rho e^{i\phi} \in \Delta$, we set $R = 2\rho$, then (9.17) gives

$$U(\rho e^{i\phi}) \leqslant o(\rho^{\lambda+1}) \qquad (\rho e^{i\phi} \in \Delta,\ \rho \to \infty). \tag{9.18}$$

For the case $\lambda = 0$ we can do better from (9.17); let $R \to \infty$ while ρ is fixed to obtain

$$U(\rho e^{i\phi}) \leqslant M \qquad (\rho e^{i\phi} \in \Delta,\ \lambda = 0). \tag{9.19}$$

Finally, (9.4) and (9.5) are obviously equivalent to (9.18) and (9.19).

Now we need the following theorem.

THEOREM 21. *Let $n \geqslant 2$ and let $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be n disjoint Jordan arcs in $\mathcal{R}\,\zeta > 0$ which join $\zeta = 0$ to $\zeta = \infty$. Let Γ_i and Γ_{i+1} bound the subdomain Δ_i of $\mathcal{R}\zeta > 0$, $i = 1, 2, \cdots, n$-1 with the Δ_i disjoint. Let Δ be the domain bounded by Γ_1 and Γ_n, which includes all Δ_i. Let $F(\zeta)$ be holomorphic in Δ, continuous at finite points of Γ_1*

and Γ_n, *and let* $M > 0$ *and* $\lambda > -1$ *be real constants such that*

$$\log |F(\zeta)| \leqslant o(\rho^{\lambda+1}) \qquad (\rho = |\zeta| \to \infty, \zeta \in \Delta) \tag{9.20}$$

and

$$|F(\zeta)| \leqslant M \qquad (\zeta \in \Gamma_i, i = 1, 2, \cdots, n). \tag{9.21}$$

If $F(\zeta)$ *is unbounded in every* Δ_i, $1 \leqslant i \leqslant n-1$, *then*

$$\lambda > n - 2. \tag{9.22}$$

Theorem 21 is the analog for a subdomain of the half-plane of Ahlfors' theorem confirming the conjecture of Denjoy on the number of asymptotic values of entire functions. The proof of Theorem 21 follows precisely the proof given by Ahlfors [1] and there is no need to repeat it.

The theorem at which this section is aimed is the following.

THEOREM 22. *Let* $f(z)$ *be holomorphic and non-constant in* $\{ |z| < 1 \}$ *and let* $M(r)$ *denote its maximum modulus. Also let, for a constant* λ, $0 \leqslant \lambda \leqslant 1$,

$$\int_0^1 (1-r)^\lambda \log M(r)\, dr < \infty. \tag{9.23}$$

Let z_0, $|z_0| = 1$, *be given and let* n_f (n_∞) *denote the number of distinct asymptotic tracts of* f *associated with* z_0 *for which the asymptotic values are finite (infinite). Then*

$$\lambda = 0 \Longrightarrow n_f \leqslant 1, n_\infty \leqslant 2 \tag{9.24}$$

$$0 < \lambda \leqslant 1 \Longrightarrow n_f \leqslant 2, n_\infty \leqslant 3. \tag{9.25}$$

Remarks. Before proving this theorem we must point out that it is closely related to results of Gehring and Heins. *Let* $f(z)$ *be holomorphic and of bounded characteristic in* $\{ |z| < 1 \}$. Let z_0, n_f, and n_∞ be as in the statement of Theorem 22. *Gehring* [23,

Theorem 1] proved that $n_f \leqslant 2$. *Heins* [24, § 9] *proved that* $n_f \leqslant 2$ *and* $n_\infty \leqslant 3$. These results are special cases of (9.25). If $T(r)$ is the characteristic of f, then as is well-known [18, p. 220], $(\rho - r) \log M(r) \leqslant (\rho + r)\ T(\rho)$ for $0 < r < \rho < 1$. Since $T(\rho) = O(1)$ it follows that $(1-r) \log M(r) = O(1)$ and (9.23) is valid for any $\lambda > 0$. It is clear however that (9.25) covers more than functions of bounded characteristic.

Gehring also considered the following subclass N^* of holomorphic functions of bounded characteristic: $f \in N^*$ *if and only if* $\log^+ |f(z)|$ *has a harmonic majorant in* $\{ |z| < 1 \}$ *which can be written as a Poisson integral. He proved* [23, Theorem 2] *that if* $f \in N^*$, *then* $n_f \leqslant 1$. This result overlaps (9.24); but neither result contains the other. It is easily seen that if f has an arc-tract (with asymptotic value ∞), then $\log^+ |f(z)|$ has no finite harmonic majorant; hence f is not of bounded characteristic, and, a fortiori, does not belong to N^*. Example 3 in section 10 shows that there are such functions which satisfy (9.23) with $\lambda = 0$. For instance, choose $\mu(r) = \exp\{(1-r)^{-1/2}\}$. Thus (9.24) is not contained in Gehring's result. On the other hand, set

$$Q(t) = |t|^{-1} (\log (2\pi/|t|))^{-2} > 0$$

for $0 < |t| \leqslant \pi$, and

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} Q(t)\ dt \qquad (z = re^{i\theta}),$$

which is convergent in $\{ |z| < 1 \}$. Also $u(z) > 0$ in $\{ |z| < 1 \}$. Let $g(z)$ be a function, holomorphic in $\{ |z| < 1 \}$, with real part $u(z)$; set $f(z) = \exp g(z)$. Then $u(z) = \log |f(z)| = \log^+ |f(z)|$, and hence $f \in N^*$. But $\log M(r) = u(r)$, and by a change in the order of integration

$$\int_0^1 \log M(r)\, dr = \frac{1}{\pi} \int_0^{\pi} Q(t)\ dt \int_0^1 \frac{1-r^2}{1-2r\cos t + r^2} dr.$$

The inner integral on the right is easily computed and is found to be $= 2 \log (2\pi/t) + O(1)$ as $t \downarrow 0$. Thus (9.23) is divergent for $\lambda = 0$, and Gehring's result is not contained in (9.24).

Proof. It should be noted that, since $\lambda \leqslant 1$, f certainly satisfies

(III) and by Theorem 14 and (7.30) and (7.31) $f \in \mathcal{A}$. In computing n_∞, one counts the occurrence of an arc tract for which $z_0 \in K$; by Theorem 3 that does not upset the following argument. To prove Theorem 22 we use Theorem 20 with $u(z) = \log|f(z)|$ and $p(r) = \log^+ M(r)$. If $\lambda = 0$, then we don't need Theorem 21. For if f has the finite asymptotic values a_1 and a_2 on C_1 and C_2 (C_1 and C_2 as in Theorem 20), it follows from (9.5) and Lindelöf's theorem [11] that $a_1 = a_2$, and C_1 and C_2 belong to the same asymptotic tract. Hence $n_f \leqslant 1$. If $n_\infty \geqslant 3$, then we could choose, by (2.8), C_1 and C_2 such that (9.2) is satisfied but such that D contains a curve on which $f \to \infty$. But this would contradict (9.5) and hence $n_\infty \leqslant 2$.

Now assume $0 < \lambda \leqslant 1$. Let, for $z_0 = 1$, $C_1, C_2, \cdots, C_n$ be disjoint curves to $z = 1$ on which $|f| \leqslant M$ such that f is *unbounded* in between each pair of C's. Map $\{|z| < 1\}$ onto $\mathcal{R}\zeta > 0$ by $\zeta = (1+z)/(1-z)$. Let C_i correspond to Γ_i and set $f(z) = F(\zeta)$. By (9.4) it follows that (9.20) holds in the domains Δ_i between the Γ's. From (9.22), $n < \lambda + 2 \leqslant 3$, and hence $n \leqslant 2$. Then (9.25) follows simply from this result, (2.6), and (2.8). This completes the proof of Theorem 22.

For examples to show that Theorem 22 is best possible see Examples 13 and 14 in section 10.

10. Examples.

EXAMPLE 1. *The modular function.* Let $w = f(z)$ map $\{|z| < 1\}$ onto the universal covering surface S of the w-sphere with the three points $\{0, 1, \infty\}$ removed. Since f omits three values, $f \in \mathcal{N}$ and by Theorem 17 $f \in \mathcal{A}$. Moreover, as is well-known, $\log M(r) = O((1-r)^{-1})$ and f satisfies (III). Also known: f has only a countable number of asymptotic values and these occur at the vertices of the fundamental triangles. That is easily seen, as follows. From the nature of S it is clear that f will have a countable number of asymptotic tracts for each of the values 0, 1, ∞ and no others. These will all be point-tracts by Theorems 4 and either 7 or 17. In particular, this example shows that *the set A may be nothing more than countable dense* even if $f \in \mathcal{N}$ and satisfies (III). *The hypothesis $A_\infty \cap \gamma = \emptyset$ in Theorem 11 is essential.*

Each component of $\{z \mid |f(z)| > \lambda\}$, $\lambda > 1$, is the interior of a Jordan curve which is tangent to $\{|z| = 1\}$ at just one point. Hence it is not difficult to see that $B^* = \{|z| = 1\}$ (see beginning of section 3), and *in general the inclusion $A^* \subset B^*$ is proper.*

EXAMPLE 2. Let $\mu(r) > 0$ be a given increasing function on

$[0, 1)$, $\mu(r) \uparrow \infty$. It was shown in [15] that there exists a function $f(z)$ holomorphic in $\{ |z| < 1 \}$ for which $M(r) < \mu(r)$ and which has *no* radial limit. We can certainly choose $\mu(r)$ so that f satisfies (III), and hence $f \in \mathcal{A}$. *So the class $\mathcal{A}$ cannot be attacked through a consideration of radial limits.* This example shows that *the inclusion $\mathcal{N} \subset \mathcal{A}$ of Theorem 17 is proper.* It also indicates why the proof of Theorem 10 is not trivial.

EXAMPLE 3. Let $\mu(r)$ be given as in Example 2. It is known, Bagemihl, Erdös and Seidel [3, Theorems 3 and 5], MacLane [14, Theorem 4; 16], that there exists $f(z)$, holomorphic in $\{ |z| < 1 \}$, such that

$$\min_{z \in J_n} |f(z)| \to \infty \qquad (n \to \infty)$$

where $\{ J_n \}$ is an expanding sequence of Jordan curves tending to $\{ |z| = 1 \}$, and such that $M(r) < \mu(r)$. We choose μ so that f satisfies (III), and hence $f \in \mathcal{A}$. Then by Theorem 3 *f has a global tract and $A_\infty = \{ |z| = 1 \}$*. In particular, *$f \in \mathcal{A}$ may have just one asymptotic tract.*

If we look at f in the half-disc $D = \{ |z| < 1 \} \cap \{ \mathcal{R} z > 0 \}$ and map D onto $\{ |w| < 1 \}$, then we obtain a function $F(w) \in \mathcal{A}$ which has *an arc-tract whose end K is not the whole circumference.*

If we choose $\mu(r) = (1-r)^{-1}$ and consider $g(z) = (1-z)^2 f(z)$, then clearly *$g \in \mathcal{A}$, g has an arc-tract with end $K = \{ |z| = 1 \}$, but no global tract,* since $g(x) \to 0$ as $x \uparrow 1$.

EXAMPLE 4. Now the condition (I) covers a large class of functions, but by no means all functions in $\mathcal{A}$. The purpose of the following example is to construct *an $f \in \mathcal{A}$ for which the integral $\sigma(\theta)$ in condition* (I) *is divergent for every θ and such that $M(r)$ grows arbitrarily rapidly.* We use Theorem 19.

Let $\{ A_n \}_1^\infty$ be a given sequence of positive numbers and set

$$(10.1) \qquad r_n = 1 - 4^{-n}, \ \rho_n = 1 - \tfrac{1}{2} \cdot 4^{-n} \qquad (n \geqslant 1).$$

Then $0 < r_1 < \rho_1 < r_2 < \rho_2 < \cdots$. Set $a_1 = \lambda_1 = 1$ and choose an increasing sequence of positive integers λ_n such that

$$(10.2) \qquad \frac{1}{3} 2^{-n} \left(\frac{r_{n+1}}{\rho_n} \right)^{\lambda_{n+1}} \geqslant A_n \qquad (n \geqslant 1),$$

$$\lambda_{n+1} \geqslant 4\,\lambda_n \qquad (n \geqslant 1), \tag{10.3}$$

and

$$\left(\frac{r_{n+1}}{\rho_n}\right)^{\lambda_{n+1}} \geqslant 2^{n+1}\left\{1 + \sum_{\nu=2}^{n} 2^{-\nu+1}\, \rho_{\nu-1}^{-\lambda_\nu}\right\} \qquad (n \geqslant 1). \tag{10.4}$$

That such a choice is possible is trivial by induction on n; at each stage it is only necessary to choose λ_{n+1} large enough to satisfy all three conditions. Define

$$a_{n+1} = 2^{-n}\, \rho_n^{-\lambda_{n+1}} \qquad (n \geqslant 1). \tag{10.5}$$

Then (10.4) is equivalent to

$$a_n\, r_n^{\lambda_n} \geqslant 2\sum_{\nu=1}^{n-1} a_\nu \qquad (n \geqslant 2), \tag{10.6}$$

and in particular it follows that

$$a_n \geqslant 2a_{n-1} \qquad (n \geqslant 2). \tag{10.7}$$

Define

$$f(z) = \sum_{n=1}^{\infty} a_n\, z^{\lambda_n}. \tag{10.8}$$

For $|\,z\,| \leqslant \rho_n$, $n \geqslant 1$, we get from (10.5)

$$\left|\sum_{\nu=n+1}^{\infty} a_\nu\, z^{\lambda_\nu}\right| \leqslant \sum_{n+1}^{\infty} a_\nu\, \rho_n^{\lambda_\nu} = \sum_{n+1}^{\infty} 2^{-\nu+1}\left(\frac{\rho_n}{\rho_{\nu-1}}\right)^{\lambda_\nu}$$

$$< \sum_{n+1}^{\infty} 2^{-\nu+1} = 2^{-n+1}, \tag{10.9}$$

which proves that (10.8) converges in $\{\,|\,z\,| \leqslant \rho_n\,\}$ and hence in

$\{ |z| < 1 \}$. From (10.3) and Theorem 19 it follows that $f \in \mathcal{A}$.

For $r_n \leqslant |z| = r \leqslant \rho_n$, $n > 1$, we have from (10.9)

$$|f(z) - a_n z^{\lambda_n}| \leqslant \sum_{1}^{n-1} a_\nu + 2^{-n+1}$$

and by (10.6),

$$|f(z)| \geqslant a_n r_n^{\lambda_n} - \sum_{1}^{n-1} a_\nu - 2^{-n+1}$$

$$\geqslant \tfrac{1}{2} a_n r_n^{\lambda_n} - 2^{-n+1}.$$

From (10.6) and (10.7) the first term on the right tends to ∞, and hence

$$|f(z)| \geqslant \frac{1}{3} a_n r_n^{\lambda_n}$$

for $n \geqslant n_0$. Or, from (10.5) and (10.2)

$$(10.10) \quad |f(z)| \geqslant \frac{1}{3} 2^{-n+1} \left(\frac{r_n}{\rho_{n-1}}\right)^{\lambda_n} \geqslant A_{n-1} \; (n \geqslant n_0, r_n \leqslant |z| \leqslant \rho_n).$$

The sequence $A_n > 0$ may be anything we wish. Given $\mu(r) > 0$ and increasing on $[0, 1)$, choose $A_{n-1} > \mu(r_{n+1})$, $n \geqslant 2$. Then it follows from (10.10) that $M(r) > \mu(r)$ for $r_{n_0} \leqslant r < 1$. Hence $M(r)$ *may grow arbitrarily rapidly even though* $f \in \mathcal{A}$.

On the other hand, we get from (10.10)

$$\int_{r_n}^{\rho_n} (1 - r) \log^+ | f(re^{i\theta}) | \, dr \geqslant 4^{-2n-1} \log A_{n-1}$$

and if we choose $\log A_{n-1} \geqslant 4^{2n}$ then it is clear that *the integral* $\sigma(\theta)$ *in condition* (I) *diverges for every* θ.

EXAMPLE 5. Now if $A_n \to \infty$, then the function of Example 4 has a global tract by (10.10) and Theorem 3. But *the gap series of Theorem 19 need not have any arc-tracts*, as is shown by the following example. Here we are concerned with the case where (8.11) is satisfied, of course. Set

$$f(z) = \sum_{1}^{\infty} a_n z^{\lambda_n}$$

where $a_n = (-)^n/n$ and $\lambda_n = 4^n$. Consider $\theta = q\pi/4^m$ where q and m are arbitrary positive integers. Then for the real part of f

$$u(re^{i\theta}) = \sum_{1}^{\infty} \frac{(-)^n}{n} r^{4^n} \cos q\pi 4^{n-m}$$

$$= \sum_{1}^{m} \frac{(-)^n}{n} r^{4^n} \cos q\pi 4^{n-m} + \sum_{m+1}^{\infty} \frac{(-)^n}{n} r^{4^n}$$

$$\to \sum_{1}^{m} \frac{(-)^n}{n} \cos q\pi 4^{n-m} + \sum_{m+1}^{\infty} \frac{(-)^n}{n} \qquad (r \uparrow 1).$$

For the imaginary part of f

$$v(re^{i\theta}) = \sum_{1}^{\infty} \frac{(-)^n}{n} r^{4^n} \sin q\pi 4^{n-m} \to \sum_{1}^{m-1} \frac{(-)^n}{n} \sin q\pi 4^{n-m} \qquad (r \uparrow 1).$$

Hence the radial limit $f(e^{iq\pi/4^m})$ exists and is finite. Thus f has no arc-tract.

EXAMPLE 6. We shall construct *a function for which both A_0 and A_∞ are dense on $\{ |z| = 1 \}$ and for which $M(r)$ may grow either arbitrarily slowly or arbitrarily fast.* Set

$$g(z) = u(z) + iv(z) = \sum_{1}^{\infty} a_n z^{\lambda_n} \qquad (|z| < 1) \tag{10.11}$$

where $\lambda_{n+1}/\lambda_n \geqslant 4$, $a_n > 0$, and $\Sigma\, a_n = \infty$. Set $f(z) = e^{g(z)}$. Then by (8.12) f has the radial limit ∞ on one dense set of radii and the radial limit 0 on another dense set of radii. So certainly $f \in \mathcal{A}$, and also f satisfies (I). Let $\mu(r) > 1$ be a given function on $[0, 1)$ such that $\mu(r) \uparrow \infty$. To achieve

$$M(r) = f(r) = e^{g(r)} < \mu(r), \tag{10.12}$$

pick $a_n = 1$ and choose the λ_n large enough so that

$$r^{\lambda_n} < 2^{-n} \log \mu(r) \qquad (0 \leqslant r < 1, n \geqslant 1)\,;$$

then (10.12) is a trivial consequence. To achieve

$$M(r) = e^{g(r)} > \mu(r), \tag{10.13}$$

choose $\lambda_n = 4^n$ and $a_n = r_n^{-\lambda_n}$ where $r_n \uparrow 1$. The r_n may be chosen so that $\log \mu(r) < n$ in $0 \leqslant r \leqslant r_{n+1}$ and (10.13) is a simple consequence: $r_n \leqslant r \leqslant r_{n+1} \Longrightarrow$

$$g(r) \geqslant \sum_{1}^{n} 1 = n > \log \mu(r).$$

It is to be noted that *the growth of $M(r)$ has nothing to do with the existence of arc-tracts for $f \in \mathcal{A}$.* In Example 6, f has no arc-tracts and $M(r)$ can be either arbitrarily small or arbitrarily large. In Example 4, f has an arc-tract with arbitrarily large $M(r)$. In Example 3, f has an arc-tract with arbitrarily small $M(r)$.

EXAMPLE 7. Here we construct an example of *a function f which satisfies* (I) *but not much more for* meas $\Theta = 0$. That is, if $\sigma(\theta)$ is defined as in (7.28), then $\sigma(\theta) = \infty$ for almost all θ. From (7.29) it follows that (II) $\Longrightarrow$ $\sigma(\theta) < \infty$ a.e. Hence *this example shows that* (7.30), (II) $\Longrightarrow$ (I), *can not be reversed.*

Let $\{ r_n \}_1^\infty$ and $\{ \rho_n \}_1^\infty$ be two given sequences such that

$$(10.14)\qquad 0 < r_1 < \rho_1 < r_2 < \rho_2 < \cdots < r_n < \rho_n < r_{n+1} < \cdots \uparrow 1.$$

We then pick a sequence $\{ a_n, \lambda_n \}_1^\infty$ with

$$(10.15)\qquad a_n > 0 \qquad (n \geqslant 1)$$

and

$$(10.16)\qquad \lambda_n = 2^{\mu_n},\ \mu_n \textit{ an increasing sequence of positive integers}$$

such that

$$(10.17)\qquad a_1 = 1, \lambda_1 = 2,$$

$$(10.18)\qquad a_n \rho_{n-1}^{\lambda_n} = 2^{-n} \qquad (n > 1)$$

and

$$(10.19)\qquad a_n r_n^{\lambda_n} (\rho_n - r_n)(1 - \rho_n) \geqslant 4 (a_1 + \cdots + a_{n-1}) \qquad (n > 1),$$

We show by induction that such a choice is possible. Now (10.18) and (10.19) are equivalent to

$$(10.18')\qquad a_n = 2^{-n} \rho_{n-1}^{-\lambda_n} \qquad (n > 1),$$

and

$$(10.19')\qquad \left(\frac{r_n}{\rho_{n-1}}\right)^{\lambda_n} \geqslant \frac{2^{n+2}}{(\rho_n - r_n)(1-\rho_n)} (a_1 + \cdots + a_{n-1}) \qquad (n > 1).$$

The choice of a_1 and λ_1 is specified by (10.17). Pick λ_2 to satisfy (10.19′) for $n = 2$ and (10.16), which is possible since $r_2 > \rho_1$. Then a_2 is determined by (10.18′). Now suppose $a_1, \lambda_1, \cdots, a_{p-1}, \lambda_{p-1}$ have been picked so that (10.18′) and (10.19′) are satisfied for $1 < n < p$. Then choose a power of 2,

$$\lambda_p = 2^{\mu_p} > 2^{\mu_{p-1}},$$

such that (10.19′) is correct with $n = p$; this is possible since $r_p > \rho_{p-1}$. Then a_p is determined by (10.18′). We note the simple consequence of (10.19)

$$a_n > 4(a_1 + \cdots + a_{n-1}) \geqslant 4a_{n-1} \qquad (n > 1).$$

But $a_1 = 1$, and it follows that

$$a_n \geqslant 4^{n-1} \qquad (n \geqslant 1). \tag{10.20}$$

Now consider the function

$$g(z) = -i \sum_{1}^{\infty} a_n z^{\lambda_n} \tag{10.21}$$

$$u(z) = \mathcal{R} g(z) = \sum_{1}^{\infty} a_n r^{\lambda_n} \sin \lambda_n \theta \tag{10.22}$$

where $z = re^{i\theta}$. First, for $|z| \leqslant \rho_m$ we obtain from (10.18)

$$\sum_{1}^{\infty} |a_n z^{\lambda_n}| \leqslant \sum_{1}^{m} a_n \rho_m{}^{\lambda_n} + \sum_{m+1}^{\infty} a_n \rho_{n-1}^{\lambda_n}$$

$$\leqslant \sum_{1}^{m} a_n + \sum_{m+1}^{\infty} 2^{-n} = \sum_{1}^{m} a_n + 2^{-m}.$$

Since $\rho_m \uparrow 1$, we see that the Taylor series (10.21) converges in $\{ |z| < 1 \}$.

Secondly, we note that if $\theta = k\pi/2^p$, where p and k are any integers, $p > 0$, then all the terms in (10.22) for which $\lambda_n = 2^{\mu_n} \geqslant 2^p$ vanish. Thus $u(z)$ is bounded on any such radius:

$$|u(z)| \leqslant M_p = \sum_{\mu_n < p} a_n \qquad (\theta = k\pi/2^p). \tag{10.23}$$

Thirdly, for $n > 1$, $r_n \leqslant |z| \leqslant \rho_n$ and θ such that $\sin \lambda_n \theta \geqslant \frac{1}{2}$, we obtain from (10.18) and (10.19)

$$u(z) \geqslant \tfrac{1}{2}\, a_n r_n^{\lambda_n} - \sum_{1}^{n-1} a_\nu - \sum_{n+1}^{\infty} a_\nu \rho_{\nu-1}^{\lambda_\nu}$$

$$\geqslant \tfrac{1}{2}\, a_n r_n^{\lambda_n} - \tfrac{1}{4}\, a_n r_n^{\lambda_n} (\rho_n - r_n)(1-\rho_n) - \sum_{n+1}^{\infty} 2^{-\nu}$$

$$> \tfrac{1}{4}\, a_n r_n^{\lambda_n} - 2^{-n}.$$

It follows from (10.19) and (10.20) that $a_n r_n^{\lambda_n} \to \infty$ as $n \to \infty$, and hence

$$(10.24)\quad u(re^{i\theta}) > \frac{1}{8} a_n r_n^{\lambda_n} \qquad (r_n \leqslant |z| \leqslant \rho_n,\ \sin \lambda_n\theta \geqslant \tfrac{1}{2},\ n \geqslant n_0).$$

Now define $f(z)$ by

$$(10.25)\qquad f(z) = e^{g(z)}.$$

Then $|f(z)| = e^{u(z)}$, and from (10.23) we see that for integers k and p, $p > 0$,

$$(10.26)\qquad |f(re^{i\theta})| \leqslant e^{M_p} \qquad (\theta = k\pi/2^p,\ 0 \leqslant r < 1).$$

Hence $f(z)$ is bounded on each one of a dense set of radii and certainly satisfies condition (I).

Now let E_n be the subset of $[0, 2\pi]$ on which $\sin \lambda_n\theta \geqslant \tfrac{1}{2}$. If $\theta \in E_n$ and $n \geqslant n_0$, then from (10.24), (10.19) and (10.20) we get

$$(10.27)\quad \int_{r_n}^{\rho_n} (1-r) \log^+ |f(re^{i\theta})|\, dr \geqslant \frac{1}{8} a_n r_n^{\lambda_n} (1-\rho_n)(\rho_n - r_n)$$

$$\geqslant \tfrac{1}{2}(a_1 + \cdots + a_{n-1}) \geqslant \tfrac{1}{2}\, a_{n-1} \geqslant \tfrac{1}{2}\cdot 4^{n-2}.$$

Set

$$(10.28)\qquad S = \limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{\nu=k}^{\infty} E_\nu.$$

Now if $\theta \in S$ (i.e., if $\theta \in E_n$ for infinitely many n) the integral $\sigma(\theta)$ of (7.28) will diverge, since it will involve infinitely many contributions of the form (10.27) from non-overlapping intervals (r_n, ρ_n). That is,

$$\sigma(\theta) = \infty \qquad (\theta \in S). \tag{10.29}$$

We wish to show that (10.29) holds almost everywhere; so we shall prove $m(S) = 2\pi$. The complement of S is

$$S' = \liminf E'_n = \bigcup_{k=1}^{\infty} \bigcap_{\nu=k}^{\infty} E'_\nu. \tag{10.30}$$

Now $\sin \lambda_n \theta = \sin 2^{\mu_n} \theta$ has period $2\pi/\lambda_n$, and E_n consists of intervals of length $\frac{1}{3}\, 2\pi/\lambda_n$ separated by intervals of E'_n of length $\frac{2}{3}\, 2\pi/\lambda_n$. Because of the regular spacing of the intervals of E'_n and E_n, it is clear that the following argument is valid. Given k_1, then

$$m(E'_{k_1} \cap E'_k) \to \frac{2}{3} m(E'_{k_1}) \qquad (k \to \infty).$$

Hence, pick $k_2 > k_1$ such that

$$m(E'_{k_1} \cap E'_{k_2}) < \frac{3}{4} m(E'_{k_1}).$$

Continuing in this fashion, we pick $k_3, k_4, \cdots$ such that

$$m(E'_{k_1} \cap \cdots \cap E'_{k_\nu}) < \frac{3}{4} m(E'_{k_1} \cap \cdots \cap E'_{k_{\nu-1}}) \qquad (\nu > 1).$$

Then it follows readily that

$$m\left(\bigcap_{\nu=1}^{\infty} E'_{k_\nu} \right) = 0,$$

and hence

$$m\left(\bigcap_{\nu=k_1}^{\infty} E'_\nu \right) = 0$$

for arbitrary k_1. Thus from (10.30) we get $m(S') = 0$ and $m(S) = 2\pi$.

EXAMPLE 8. Here we give an example to show that *the implication* (III) $\Rightarrow$ (II) *can not be reversed.* Let $\beta > 0$ and consider the function

$$f(z) = e^{(1-z)^{-\beta}} \qquad (|z| < 1) \tag{10.31}$$

where the determination of $(1-z)^{-\beta}$ in $\{ |z| < 1 \}$ is that which has the value 1 at $z = 0$. Thus if $z = re^{i\theta}$ and $1 - z = \rho e^{i\phi}$,

$$(1-z)^{-\beta} = \rho^{-\beta} e^{-i\beta\phi}, \qquad |\phi| < \pi/2.$$

Then

$$|f(z)| = \exp\{ \rho^{-\beta} \cos \beta\phi \}, \tag{10.32}$$

and it is clear that

$$\log M(r) = (1-r)^{-\beta}. \tag{10.33}$$

Hence *f satisfies* (III) *if and only if*

$$\beta < 2. \tag{10.34}$$

From (10.32)

$$m(r) \leqslant \frac{1}{\pi} \int_0^\pi \rho^{-\beta} d\theta. \tag{10.35}$$

But $\rho^2 = (1-r)^2 + 4r \sin^2 \dfrac{\theta}{2} \geqslant (1-r)^2 + 4r(\theta/\pi)^2$

$$\geqslant \begin{cases} (1-r)^2 & (0 \leqslant \theta \leqslant 1-r) \\ 4r(\theta/\pi)^2 & (1-r \leqslant \theta \leqslant \pi). \end{cases}$$

It follows from these estimates and (10.35) that

$$m(r) = \left\{ \begin{array}{ll} O[(1-r)^{1-\beta}] + O(1) & (\beta \neq 1) \\ O\left[\log \dfrac{1}{1-r} \right] & (\beta = 1) \end{array} \right\} \quad (r \uparrow 1). \tag{10.36}$$

On the other hand, $m(r) = T(r)$ and [18, p. 220]

$$\log M(\sigma) \leqslant \frac{r+\sigma}{r-\sigma} m(r) \qquad (0 < \sigma < r < 1).$$

Setting $\sigma = 1 - 2(1-r)$ and using (10.33)

$$m(r) > 2^{-1-\beta}(1-r)^{1-\beta} \qquad (\tfrac{1}{2} \leqslant r < 1). \tag{10.37}$$

From (10.36) and (10.37) we see that *f satisfies* (II) *if and only if*

$$\beta < 3. \tag{10.38}$$

Comparing (10.34) and (10.38), we see that *f satisfies* (II) *but not* (III) *if* $2 \leqslant \beta < 3$.

Also, using $\beta \geqslant 3$, we have an example of *a function f which certainly satisfies* (I), *for f is holomorphic at all except one point of* $\{ |z| = 1 \}$, *but not* (II). These f behave quite differently from those in Example 7.

EXAMPLE 9. This example complements Example 7. We shall construct *an f for which* $\sigma(\theta) < \infty$ *on a set of measure* 2π *and* $\sigma(\theta) = \infty$ *on a dense set.*

Let

$$\phi(z,\alpha) \equiv (1 - e^{-i\alpha}z)^{-2}. \tag{10.39}$$

If $z = re^{i(\alpha+t)}$, then

$$\mathcal{R}\,\phi(z,\alpha) = \frac{1 - 2r\cos t + r^2\cos 2t}{(1 - 2r\cos t + r^2)^2}.$$

For $t = 0$ we get, for a given r, the maximum of both $|\phi|$ and $\mathcal{R}\,\phi(z,\alpha)$:

$$M_\phi(r) = \mathcal{R}\,\phi(re^{i\alpha}, \alpha) = (1-r)^{-2}. \tag{10.40}$$

Otherwise,

$$|\mathcal{R}\,\phi(z,\alpha)| \leqslant \frac{4}{(1 - 2r\cos t + r^2)^2}. \tag{10.41}$$

Now $1-2r\cos t+r^2=(1-r)^2+4r\sin^2(t/2)$, and by considering $0\leqslant r\leqslant \frac{1}{2}$ and $\frac{1}{2}\leqslant r\leqslant 1$ it is easily seen that there is a positive constant c' such that

$$1-2r\cos t+r^2\geqslant c'\,t^2 \qquad (0\leqslant r<1,\ |\,t\,|\leqslant\pi). \tag{10.42}$$

Thus

$$|\,\mathcal{R}\,\phi(re^{i\theta},\alpha)\,|\leqslant c\,|\,\theta-\alpha\,|^{-4} \tag{10.43}$$

where $\theta=\arg z$ is chosen so that $|\,\theta-\alpha\,|$ is minimal and $c=4c'^{-2}$.

Now let $\{\,\alpha_n\,\}_1^\infty$ be a given sequence of distinct points in $[0,2\pi)$ which is everywhere dense in $[0,2\pi)$. We shall use the function

$$\psi(z)=\sum_1^\infty \delta_n\,\phi(z,\alpha_n) \tag{10.44}$$

for a suitable choice of $\delta_n>0$. By (10.40) this series converges and ψ is holomorphic in $\{\,|\,z\,|<1\,\}$ if

$$\Sigma\,\delta_n<\infty. \tag{10.45}$$

Now choose $\{\,\eta_n\,\}_1^\infty$, $\eta_n>0$, such that

$$\eta_n<2^{-n} \tag{10.46}$$

and

$$\left\{\begin{array}{l}\textit{The sector } |\arg z-\alpha_n\,|\leqslant\eta_n \textit{ contains none of}\\ \textit{the radii } \arg z=\alpha_\nu,\ 1\leqslant\nu<n.\end{array}\right. \tag{10.47}$$

That is possible since the α_n are distinct. From (10.46) it follows that

$$\sum_1^\infty \eta_n<1. \tag{10.48}$$

Now choose δ_n so that

$$0 < \delta_n \leqslant c^{-1} 2^{-n} \eta_n{}^4. \tag{10.49}$$

By virtue of (10.46), (10.45) is a consequence of (10.49) and (10.49) is the only condition on the δ_n.

Now from (10.40)

$$\mathcal{R}\,\psi(re^{i\alpha_n}) = \delta_n(1-r)^{-2} + \mathcal{R} \sum_{\nu=1}^{n-1} \delta_\nu \phi(re^{i\alpha_n}, \alpha_\nu)$$

$$+ \sum_{\nu=n+1}^{\infty} \delta_\nu\, \mathcal{R}\, \phi(re^{i\alpha_n}, \alpha_\nu).$$

For the middle term on the right each summand is bounded and the term is bounded. For the third term use (10.43) and $|\alpha_n - \alpha_\nu| > \eta_\nu$ from (10.47). Thus we obtain

$$\mathcal{R}\,\psi(re^{i\alpha_n}) \geqslant \delta_n(1-r)^{-2} - C_n - \sum_{\nu=n+1}^{\infty} \delta_\nu c\, \eta^{-4}{}_\nu$$

where C_n is a constant. Finally, by (10.49)

$$\mathcal{R}\,\psi(re^{i\alpha_n}) \geqslant \delta_n(1-r)^{-2} - C_n - 1 \qquad (n \geqslant 1,\ 0 \leqslant r < 1). \tag{10.50}$$

Now let I_n denote the interval $\{\,|\theta - \alpha_n| < \eta_n\,\}$; let

$$E_n = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\} \cup \bigcup_{\nu=n}^{\infty} I_\nu, \tag{10.51}$$

and let E'_n be the complement of E_n relative to $[0, 2\pi)$.

We note from (10.46) that

$$m(E_n) \leqslant \sum_{\nu=n}^{\infty} 2\eta_\nu < 2^{-n+2},$$

and hence

$$m(E'_n) > 2\pi - 2^{-n+2}. \tag{10.52}$$

For a fixed $\theta \in E'_n$, $z = re^{i\theta}$,

$$|\mathcal{R}\,\psi(z)| \leqslant \sum_{\nu=1}^{n-1} \delta_\nu\,|\mathcal{R}\,\phi(z,\alpha_\nu)| + \sum_{\nu=n}^{\infty} \delta_\nu\,|\mathcal{R}\,\phi(z,\alpha_\nu)|.$$

Each term in the first sum is bounded by (10.51). For the second sum we use (10.43), $|\theta - \alpha_\nu| \geqslant \eta_\nu$ (from (10.51)) and (10.49). The result is

$$|\mathcal{R}\,\psi(z)| \leqslant B_n(\theta) + \sum_{\nu=n}^{\infty} 2^{-\nu} \leqslant B_n(\theta) + 1 \qquad (\theta \in E'_n) \tag{10.53}$$

where $B_n(\theta)$ is a finite function of θ defined on E'_n.

Clearly $E_n \downarrow E$, $E'_n \uparrow E'$; from (10.52)

$$m(E') = 2\pi. \tag{10.54}$$

If $\theta \in E'$, then $\theta \in E'_n$ for some n and we get from (10.53)

$$|\mathcal{R}\,\psi(z)| < B(\theta) \qquad (\theta = \arg z \in E') \tag{10.55}$$

where $B(\theta)$ is a finite function of θ defined on E'.

Now define

$$f(z) = e^{\psi(z)}\,; \tag{10.56}$$

f is holomorphic in $\{\,|z| < 1\,\}$. From (10.50) we see that

$$\sigma(\alpha_n) = \int_0^1 (1-r)\,\log^+|f(re^{i\alpha_n})|\,dr = \infty \qquad (n \geqslant 1). \tag{10.57}$$

On the other hand, it follows from (10.55) that

$$\sigma(\theta) < \infty \qquad (\theta \in E'). \tag{10.58}$$

It will be recalled that $\{\alpha_n\}_1^\infty$ was picked to be dense on $[0, 2\pi]$. So (10.57), (10.58), and (10.54) are the properties we claimed for f at the beginning of this example.

EXAMPLE 10. The purpose of this example is to show that Theorem 11 is non-trivial in that it is not a simple consequence of Fatou's theorem. That is, *we shall construct $f \in \mathcal{A}$ such that*

$$A_\infty = \emptyset \tag{10.59}$$

but

$$\limsup_{z \to \zeta} |f(z)| = \infty \qquad (\text{all } \zeta, |\zeta| = 1). \tag{10.60}$$

Also f will be such that for any ζ, $|\zeta| = 1$, and any finite a, ζ will be a limit point of zeros of $f(z) - a$. Hence it is impossible to reduce f to a bounded function in any neighborhood of ζ by means of a linear transformation of f.

A slight alteration of the construction will give a function satisfying (10.59) and (10.60) but such that f assumes no values in a certain half-plane. Then a linear transformation of f leads to a *bounded holomorphic function F in $\{|z| < 1\}$ such that : $F \neq 0$, F does not have zero as an asymptotic value, but for any ζ, $|\zeta| = 1$,*

$$\liminf_{z \to \zeta} |F(z)| = 0. \tag{10.61}$$

About the simplest example of an unbounded function in $\{|z| < 1\}$ which does not have ∞ as an asymptotic value may be obtained as follows. Let D_0 be the domain $\{u > 0, 0 < v < e^{-u}\}$ in the $w = u+iv$-plane. Let D be the domain obtained by slitting D_0 along each of the segments $\{v = 1/n, 1 \leqslant u \leqslant \log n\}$ $(n \geqslant 3)$. Let $w = f(z)$ map $\{|z| < 1\}$ onto D. Then f is unbounded, but there is no path in D on which $w \to \infty$, and so f does not have ∞ as an asymptotic value. Our construction is a generalization of the essential fact about D: every way you start for ∞ you get stuck and have to turn back and try a different channel; a frustrating maze.

Let γ be a Jordan arc in the w-plane and let

$$\sup_{z \in \gamma} |z| = \rho(\gamma). \tag{10.62}$$

By a *piece* $S(\gamma,n)$, we shall mean a *simply-connected* Riemann surface over the w-plane with the following properties.

(10.63) *$S(\gamma,n)$ is the interior of a compact bordered surface $S(\gamma,n) \cup \Gamma(\gamma,n)$ over the w-plane where Γ is a Jordan curve on a larger Riemann surface containing $S \cup \Gamma$.*

(10.64) *$\Gamma(\gamma,n)$ contains an arc $\Gamma_0(\gamma,n)$ projecting one-one onto γ.*

(10.65) *There is a cross-cut C (γ,n) of $S(\gamma,n)$, lying over $\{ |w| < 1 \}$, which separates S into two parts S_1 and S_2 such that:*

(10.66) *S_1 has Γ_0 on its boundary and S_1 lies over $\{ |w| < \rho(\gamma) + n^{-2} \}$, and*

(10.67) *the projection of S_2 contains $\{ |w| < n \}$ and is contained in $\{ |w| < n+1 \}$.*

That such pieces always exist (given γ and n) is easily seen. Take S_1 to be a "strip" with one end bounded by γ and the other by a given segment C inside $\{ |w| < 1 \}$. This is possible if C is chosen so that $C \cap \gamma = \emptyset$. Construct S_2 by taking a "strip" S^* with one end bounded by C (S^* and S_1 abut on C from opposite sides) and the other by an arc on $\{ |w| = n + \frac{1}{2} \}$ and such that $S^* \subset \{ |w| < n+1 \}$. Put a slit in S^* starting at a point over $\{ |w| = n + \frac{1}{2} \}$ and hang on with a branch-point the disc $\{ |w| < n + \frac{1}{2} \}$ similarly slit. S^* plus this disc is S_2. Clearly, $S(\gamma,n)$ may be picked so it abuts on either side of γ desired.

We now construct a Riemann surface $\mathcal{F}$ as follows. Let $\mathcal{F}_1$ be the disc $\{ |w| < 1 \}$ and partition its boundary into arcs $\gamma_{1,\nu}$ $1 \leqslant \nu \leqslant \nu_1$. Along each $\gamma_{1,\nu}$ hang a piece $S(\gamma_{1,\nu},2)$ and call the resulting surface $\mathcal{F}_2$. Here of course all the $S(\gamma_{1,\nu},2)$ are chosen to abut on $\gamma_{1,\nu}$ from the side opposite to that on which $\mathcal{F}_1$ abuts. Partition the boundary of $\mathcal{F}_2$ into arcs $\gamma_{2,\nu}$, $1 \leqslant \nu \leqslant \nu_2$ and hang on pieces $S(\gamma_{2\nu}, 3)$ to get the surface $\mathcal{F}_3$. And so on. $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \uparrow \mathcal{F}$. Clearly, $\mathcal{F}$ is a simply-connected covering of the w-plane.

In partitioning the boundary of $\mathcal{F}_n$ into arcs $\gamma_{n,\nu}$, $1 \leqslant \nu \leqslant \nu_n$, we always choose the partition so that

(10.68) *each end point of a* $\gamma_{n-1,\nu}$ *is an end point of two* $\gamma_{n,k}$. Then the end points of all $\gamma_{n,\nu}$ are boundary points, not interior points, of $\mathcal{F}$.

Let $w = f(z)$ map $\{ \mid z \mid < R \leqslant \infty \}$ onto $\mathcal{F}$ with $f(0) = 0 \in \mathcal{F}_1$. If f had the asymptotic value ∞, then there would exist a curve Γ on $\mathcal{F}$ on which $\mid w \mid \to \infty$. Since $\mathcal{F}_n$ lies over $\{ \mid w \mid < n+1 \}$ (see (10.66) and (10.67)), Γ must run through all $\mathcal{F}_n$. Also Γ must run through infinitely many of the $S_2(\gamma_{n,\nu}, n+1)$; for if eventually Γ stayed in various S_1 then by (10.66), since $\Sigma\, n^{-2} < \infty$, w would be bounded on γ. But to get from one S_2 to another Γ must go through some S_1 and cross at least one curve $C(\gamma, n)$ (see (10.65)) on which $\mid w \mid < 1$. Hence a Γ on which $\mid w \mid \to \infty$ does not exist, and f does *not* have the asymptotic value ∞. Now any entire function does have the asymptotic value ∞ [8;18,p.291] and hence $R < \infty$; we may take $R = 1$.

Now it may be true that $f \in \mathcal{A}$ without more ado. But let $\mu(r)$ be a given positive function on $[0, 1)$ such that $\mu(r) \uparrow \infty$ as $r \uparrow 1$. Then by the use of Carathéodory's kernel theory [5], [13], it can be shown that if the partitions $\{ \gamma_{n,\nu} \}$ are chosen fine enough at successive stages n, then

$$M(r) < \mu(r) \qquad (0 \leqslant r < 1) \tag{10.69}$$

where $M(r)$ is the maximum modulus of $f(z)$. The details of the argument are lengthy; but they are similar to those given in [16] and we skip them here. It is also useful to choose the partitions $\{ \gamma_{n,\nu} \}$ fine enough so that

$$\max_{1 \leqslant \nu \leqslant \nu_n} \operatorname{diam} \gamma_{n,\nu} \to 0 \qquad (n \to \infty). \tag{10.70}$$

If now $\int_0^1 (1-r) \log^+ \mu(r)\, dr < \infty$, then by (10.69) f certainly satisfies (III) and $f \in \mathcal{A}$. By (10.68) each $\gamma_{n,\nu}$ is a cross-cut of $\mathcal{F}$; let the image of $\gamma_{n,\nu}$ in $\{ \mid z \mid < 1 \}$ be $\gamma^*_{n,\nu}$. If the end points of $\gamma_{n,\nu}$ are over $w = a_{n,\nu}$ and $b_{n,\nu}$, then f will have the finite asymptotic values $a_{n,\nu}$ and $b_{n,\nu}$ along the two ends of $\gamma^*_{n,\nu}$. Hence by Theorem 4, $\gamma^*_{n,\nu}$ tends to two definite points of $\{ \mid z \mid = 1 \}$; those points are distinct by (2.6) since f does not have the asymptotic value ∞. Thus

each $\gamma^*_{n,\nu}$ is a cross-cut of $\{\ |z| < 1\ \}$. From the structure of $\mathcal{F}_n$ and $\mathcal{F}$ it is clear that for any fixed n the $\gamma^*_{n,\nu}$ form a sequence of cross-cuts "running around" the circumference $\{\ |z| = 1\ \}$ and all together bounding the part of $\{\ |z| < 1\ \}$ corresponding to $\mathcal{F}_n$. Now

$$\max_{1 \leqslant \nu \leqslant \nu_n} \operatorname{diam} \gamma^*_{n,\nu} \to 0 \qquad (n \to \infty). \tag{10.71}$$

For if we had a sequence γ^*_{n,i_n} such that

$$\limsup_{n \to \infty} \operatorname{diam} \gamma^*_{n,i_n} > 0,$$

then by (10.70) we could pick a subsequence $\gamma^*_{n_q\ i_{n_q}}$, tending to a whole arc of $\{\ |z| = 1\ \}$, on which $f \to a$. Now $a \neq \infty$ is out by Theorem 9 applied to $f - a$. If $a = \infty$, then f would have the asymptotic value ∞ by Theorem 3, which is also impossible. Hence (10.71) holds.

Now let $|\zeta| = 1$ and let

$$U = \{\ |z| < 1\ \} \cap \{\ |z-1| < \delta\ \},$$

where $\delta > 0$ is arbitrary. Since the $\gamma^*_{n,\nu}$, for fixed n "run around" $\{\ |z| = 1\}$ and by (10.71), there exists n such that U contains a complete $\gamma^*_{n,\nu}$, say γ^*_{n,p_n}. Let Δ be that one of the two domains into which γ^*_{n,p_n} cuts $\{\ |z| < 1\ \}$ such that $\Delta \subset U$. From the structure of $\mathcal{F}$, Δ contains the images of various $S(\gamma_{m,\nu}, m+1)$ for any $m \geqslant n$. Hence by (10.67) f assumes all finite values infinitely often in Δ and hence in U. In particular, f is unbounded in U. Thus we have proved (10.59), (10.60), and the other properties stated at the beginning of Example 10.

The alteration mentioned in the second paragraph of Example 10 is the following. Replace (10.67) by: the projection of S_2 contains $\{\ |w| < n;\ \mathcal{R}w > 0\ \}$ and is contained in $\{\ |w| < n+1;\ \mathcal{R}w > -1\ \}$. It is clear then by (10.66) that $\mathcal{F}$ lies over

$$\mathcal{R}w > -1 - \sum_{2}^{\infty} n^{-2} = -\pi^2/6.$$

The rest of the argument is changed only at the very end; f takes every value in $\mathcal{R}w > 0$ infinitely often in U.

EXAMPLE 11. We wish *to show that Theorem 11 can **not** be improved to* $m(A^* \cap \gamma) = m(\gamma)$. The following example depends on the fact that there exist Cantor type sets of positive measure. Let E_1 consist of two open arcs $|\arg z| < \delta_1$ and $|\arg z - \pi| < \delta_1$ on $\{|z| = 1\}$. Let E_2 be E_1 plus two open arcs removed from E'_1: $|\arg z \pm \pi/2| < \delta_2$. E_3 is obtained by adding to E_2 four open arcs, each of length δ_3, one centered in each of the four arcs in E'_2. And so on. The δ_n are chosen so that none of the intervals in E_n overlap or have a common end point. Also define

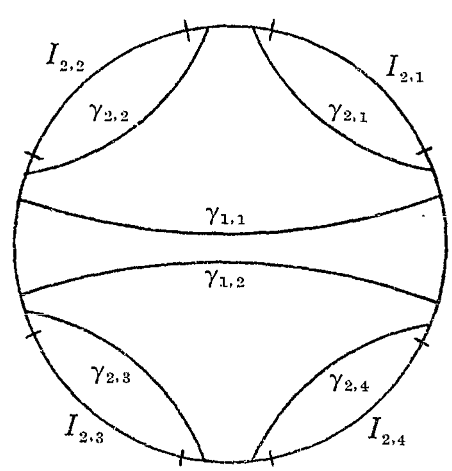
$$E = \bigcup_1^\infty E_n \qquad E' = \bigcap_1^\infty E'_n. \tag{10.72}$$

Note that

$$m(E) = 2\delta_1 + \sum_2^\infty 2^{n-1}\delta_n,$$

and we can choose the δ_n so that $m(E) < \varepsilon$. Then $m(E') > 2\pi - \varepsilon$.

For each interval $I_{n,\nu}$ of E'_n take a cross-cut $\gamma_{n,\nu}$ of $\{|z| < 1\}$ with end points in the two intervals of E_n adjacent to $I_{n,\nu}$. The γ's may be chosen for every n and ν so that no two intersect or have a com-



mon end point and so that the set of all end points has no limit point in E. Also we require that only a finite number of γ's intersect any disc $\{|z| \leqslant r\}$, $r < 1$. For each $\gamma_{n,\nu}$ let $D_{n,\nu}$ be a "strip" subdomain

of $\{ | z | < 2 \}$ between a pair of cross cuts of $\{ | z | < 2 \}$ such that $\gamma_{n,\nu} \subset D_{n,\nu}$. The four end points of the two cross-cuts bounding $D_{n,\nu}$ are to be distinct. The $D_{n,\nu}$ are chosen so that any two have disjoint closures, which is clearly possible.

Now by Runge's approximation theorem there exist polynomials $P_{n,\nu}(z)$ such that

$$| P_{n,\nu}(z) - 1 | < \tfrac{1}{4} \qquad (z \in \gamma_{n,\nu}) \tag{10.73}$$

and

$$| P_{n,\nu}(z) | < 4^{-n-1} \qquad (z \in D'_{n,\nu} \cap \{ | z | < 2 \}). \tag{10.74}$$

Then set

$$f(z) = \sum_{n=1}^{\infty} \sum_{\nu=1}^{2^n} P_{n,\nu}(z). \tag{10.75}$$

It follows from (10.74) that this series converges uniformly in some neighborhood of any point of $\{ | z | < 2 \}$ which is not a limit point of the assorted $D_{n,\nu}$. In particular *then $f(z)$ is holomorphic on $\{ | z | < 1 \} \cup E$. Certainly then f has a finite asymptotic value at each point of E.* Since E is dense it follows that $f \in \mathcal{A}$.

Suppose now that $\zeta \in E'$ and Γ is a curve in $\{ | z | < 1 \}$ from $z = 0$ to ζ. Then Γ must intersect each of a sequence of arcs $\gamma_{1,i_1}, \gamma_{2,i_2}, \cdots, \gamma_{n,i_n}, \cdots$. So there will be a sequence of points on Γ, progressing along Γ and tending to ζ, $z_1, z_1^*, z_2, z_2^*, \cdots$ such that

$$z_n \in \gamma_{n,i_n}, \qquad z_n^* \in \Big(\bigcup_{m,\nu} D_{m,\nu} \Big)'.$$

Now it follows from (10.73), (10.74), and (10.75) that

$$| f(z_n) | \geqslant | P_{n,i_n}(z_n) | - \sum_{\text{other}} | P_{m,\nu}(z_n) | > \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

and

$$| f(z_n^*) | \leqslant \sum_{m,\nu} | P_{m,\nu}(z_n^*) | < \sum_{m=1}^{\infty} 2^m \cdot 4^{-m-1} = \frac{1}{4}.$$

Hence it is clear that f has no asymptotic value at ζ; that is, $A^* \cap E' = \emptyset$. Since $m(E') > 2\pi - \varepsilon$, we see that $m(A^* \cap \gamma) > 0$ is the strongest conclusion we can draw on $m(A^* \cap \gamma)$ in Theorem 11.

However, in this example, E' is nowhere dense and hence any γ contains an interval γ_1 such that $m(A^* \cap \gamma_1) = m(\gamma_1)$. We don't know whether such a conclusion in Theorem 11 would be correct in general.

EXAMPLE 12. We recall the example of Lusin [20, p. 229] of a non-constant function $f(z)$, holomorphic in $\{ | z | < 1 \}$, which has the radial limit zero on a set of measure 2π. Clearly $f \in \mathcal{A}$. Also A_∞ is dense on $\{ | z | = 1 \}$. For if not, then as in the proof of Theorem 11, f would have asymptotic values of modulus $> n$ on a set E of positive measure. This is impossible by $m(A_0) = 2\pi$ and the theorem of Bagemihl and Heins (see section 5). It is interesting to compare this example with Examples 6 and 9.

EXAMPLE 13. We wish to show that *the estimate* (9.24) *in Theorem 22 can not be improved.* Let $\zeta = (1+z)/(1-z)$, which maps $\{ | z | < 1 \}$ onto $\{ \mathcal{R}\, \zeta > 0 \}$. If $| z | = r$ and $\rho = | \zeta |$, then

$$\rho = | \zeta | \leqslant \frac{1+r}{1-r}.$$

Take $f(z) = F(\zeta) = \zeta e^{-\zeta}$. Then

$$P(\rho) = \max_{| \zeta | = \rho,\, \mathcal{R}\, \zeta \geqslant 0} | F(\zeta) | = \rho$$

and $M(r)$, the maximum modulus of f, satisfies

$$M(r) \leqslant P\left(\frac{1+r}{1-r}\right) = \frac{1+r}{1-r} < \frac{2}{1-r}.$$

Then $\displaystyle\int_0^1 \log M(r)\, dr < \infty$. As a matter of fact, (9.23) converges for $\lambda > -1$. The function f has $n_f = 1$ and $n_\infty = 2$ at $z_0 = 1$. For $F(\xi) \to 0$ as $\xi \uparrow \infty$ and $F(1 + i\eta) \to \infty$ as $\eta \to +\infty$ or $\eta \to -\infty$. Here f has the angular limit 0 at $z_0 = 1$ and the two tracts for ∞ are squeezed toward $\{ | z | = 1 \}$.

A different behavior is exhibited by

$$f(z) = G(\zeta) = \zeta^{-1} \sin(\zeta^{\alpha})$$

where $0 < \alpha < 1$. Here $|G(\zeta)| = O[\exp(\rho^{\alpha})]$ and

$$\log M(r) = O[(1-r)^{-\alpha}]$$

so $\int_0^1 \log M(r)\, dr < \infty$. But now $G(\zeta) \to 0$ on the real axis but

$G(\zeta) \to \infty$ in each of the angles $\delta \leqslant |\arg \zeta| \leqslant \pi/2 - \delta$. So now f will have two "angular tracts" for ∞ and the tract for 0 is squeezed toward the radius $[0, 1)$.

EXAMPLE 14. We now *consider the case* $0 < \lambda \leqslant 1$ *of Theorem 22. We show that* (9.25) *can not be improved.* Given λ, choose δ, $0 < \delta < \lambda$, and set

$$\zeta = \left(\frac{1+z}{1-z}\right)^{1+\delta}$$

which maps $\{|z| < 1\}$ onto $\Delta = \{|\arg \zeta| < (1+\delta)\pi/2\}$. Then

$$\rho = |\zeta| \leqslant \left(\frac{1+r}{1-r}\right)^{1+\delta} = O[(1-r)^{-1-\delta}]. \tag{10.76}$$

Set

$$f(z) = F(\zeta) = \zeta^{-1} \sinh \zeta$$

and let $P(\rho)$ be the maximum of $|F(\zeta)|$ in $\{|\zeta| \leqslant \rho\}$. Then $P(\rho) = O(e^{\rho})$, and from (10.76)

$$\log M(r) \leqslant \log P[B(1-r)^{-1-\delta}] = O[(1-r)^{-1-\delta}].$$

Since $\delta < \lambda$,

$$\int_0^1 (1-r)^{\lambda} \log M(r)\, dr < \infty. \tag{9.23}$$

Also it is clear that $n_f = 2$ and $n_\infty = 3$ at $z_0 = 1$. For $F(\zeta) \to \infty$ along each of the three rays $\arg \zeta = 0$ and $\arg \zeta = \pm(2+\delta)\pi/4$. And those three rays are separated by the rays $\arg \zeta = \pm \pi/2$ along which $F(\zeta) \to 0$.

If we set $u(z) = \log |f(z)|$, then the hypotheses of Theorem 20 are satisfied with $p(r) = \log^+ M(r)$ and C_1 and C_2 the images of $\arg \zeta = \pm \pi/2$. But for $0 < x < 1$,

$$u(x) = \log f(x) \sim \left(\frac{1+x}{1-x}\right)^{1+\delta} \qquad (x \uparrow 1).$$

Since δ may be chosen arbitrarily close to λ, *the conclusion* (9.4) *of Theorem 20 can* **not** *be replaced by*

$$u(z) \leqslant O[\,|1-z|^{-\lambda^{-1}+\varepsilon}]$$

no matter how small $\varepsilon > 0$.

EXAMPLE 15. Set

$$\phi(\zeta) = \zeta + \frac{1}{\zeta} + \frac{1}{1-\zeta},$$

which maps the sphere onto a three sheeted surface S with three distinct points over ∞ corresponding to $\zeta = 0, 1, \infty$. Let $\mu(z)$ be the modular function of Example 1 and set $f(z) = \phi(\mu(z))$. Then $w = f(z)$ maps $\{\,|z| < 1\,\}$ onto the universal covering $\tilde{S}$ of $S - \{\infty_1, \infty_2, \infty_3\}$, where ∞_i are the three points of S over ∞. Now μ and hence f is a normal holomorphic function: $f \in \mathcal{N}$. Thus $f \in \mathcal{A}$ by Theorem 17. The only asymptotic values of μ are 0,1, ∞ and *the only asymptotic values of f are ∞*. For $0 < \varepsilon < \varepsilon_0$ the components of $\tilde{S}$ over $\{\,|w| > 1/\varepsilon\,\}$ are logarithmic-branch-point neighborhoods of ∞, countable in number. So f has a countable number of tracts, all with asymptotic value ∞. These are all point-tracts, by either Theorem 7C or by Theorem 17. Thus $A^* = \emptyset$. A_∞ is countable, and *the relation*

$$(4.1) \qquad A_\infty \supset B^{*-\prime}$$

cannot be strengthened by replacing B by A.

As in Example 1, $B^* = \{\,|z| = 1\,\}$.

EXAMPLE 16. Finally we wish to show that *the inclusion* $\mathcal{L} \subset \mathcal{L}^*$ *of Theorem 1 is proper*. Here it is a little simpler to operate in the square $S = \{ 0 < x < 1, 0 < y < 1 \}$ than in the unit circle. Let γ_n be the segment $\{ x = 1/n, 0 < y < 1 \}$, $n \geqslant 2$. Let D_n be the strip

$$\{ | x - 1/n | < \frac{1}{4} n^{-3}, -1 < y < 2 \} \qquad (n \geqslant 2).$$

Then the D_n are disjoint neighborhoods of the γ_n. Let

$\gamma'_n = \{ x = c_n, 0 < y < 1 \}$ where the constants c_n are chosen so that γ'_n lies between D_n and D_{n+1}; that is,

$$\frac{1}{n+1} + \frac{1}{4(n+1)^3} < c_n < \frac{1}{n} - \frac{1}{4n^3} \qquad (n \geqslant 2).$$

Let $R = \{ 0 < x < 2, -1 < y < 2 \}$. By Runge's approximation theorem there exist polynomials $P_n(z)$ such that

$$| P_n(z) - 2 | < \tfrac{1}{2} \qquad (z \in \gamma_n, n \geqslant 2) \tag{10.77}$$

and

$$| P_n(z) | < 2^{-n} \qquad (z \in D'_n \cap R, n \geqslant 2). \tag{10.78}$$

Set

$$f(z) = \sum_{n=2}^{\infty} P_n(z). \tag{10.79}$$

If follows from (10.78) that $f(z)$ is holomorphic at every point of R which is not a limit point of infinitely many D'_n; but that is all of R. Hence in particular $f(z)$ is holomorphic in

$$S^* = \{ 0 < x \leqslant 1, 0 \leqslant y \leqslant 1 \}.$$

Now on γ_n it follows from (10.77) and (10.78) that

$$| f(z) | > (2 - \tfrac{1}{2}) - \sum_{n=2}^{\infty} 2^{-n} = 1 \quad (z \in \gamma_n, n \geqslant 2) \tag{10.80}$$

and for γ'_n

$$(10.81) \qquad |f(z)| < \sum_{n=2}^{\infty} 2^{-n} = \tfrac{1}{2} \qquad (z \in \gamma'_n, n \geqslant 2).$$

From (10.80) and (10.81) it is clear that any given level curve $C(\lambda)$ can't intersect both a γ_n and a γ'_m and is hence restricted to some $\{\delta \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ in which f is holomorphic. Hence $C(\lambda)$ ends at points and $f \in \mathcal{L}^*$. But also from (10.80) and (10.81) the level set $L(\tfrac{3}{4})$ will contain, for each $n \geqslant 2$, a cross-cut of the square S, separating γ_n and γ'_n. Thus $L(\tfrac{3}{4})$ does not end at points and $f \notin \mathcal{L}$.

REFERENCES

1. L. Ahlfors, *Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen,* Acta Soc. Sci. Fenn. N.s.*1* (1930) No. 9.

2. F. Bagemihl, *Curvilinear cluster sets of arbitrary functions,* Proc. Nat. Acad. Sci. USA *41* (1955) 379-382.

3. F. Bagemihl, P. Erdös, and W. Seidel, *Sur quelques propriétés frontières des fonctions holomorphes défines par certains produits dans le cercle-unité,* Ann. Sci. École Norm. Sup. (3) *70* (1953) 135-147.

4. P. Boutroux, *Sur l'indétermination d'une fonction uniforme au voisinage d'une singularité transcendente,* Ann. Sci. École Norm. Sup. (3) *25* (1908) 319-370.

5. C. Carathéodory, *Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten,* Math. Ann. *72* (1912) 107-144.

6. W. Grosz, *Eine ganze Funktion, für die jede komplexe Zahl Konvergenzwert ist,* Math. Ann. *79* (1918) 201-208.

7. M. Heins, *A property of the asymptotic spots of a meromorphic function or an interior transformation whose domain is the open unit disk,* Jour. Ind. Math. Soc. *24* (1960) 265-268.

8. F. Iversen, *Recherches sur les fonctions inverses des fonctions méromorphes,* Thesis, Helsingfors 1914.

9. P. Koebe, *Abhandlungen zur Theorie der konformen Abbildung,* I, Jour. f.d. reine u. angew. Math. *145* (1915) 177-223.

10. O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions,* Acta Math. *97* (1957) 47-65.

11. E. Lindelöf, *Sur un principe général de l'analyse et ses applications a la théorie de la représentation conforme,* Acta. Soc. Sci. Fenn. *46* (1915) No. 4.

12. A. J. Lohwater and G. Piranian, *The boundary behavior of functions analytic in a disk,* Ann. Acad. Sci. Fenn. A. I. No. 239 (1957).

13. G. R. MacLane, *Riemann surfaces and asymptotic values associated with real entire functions,* Rice Institute Pamphlet, special issue, Nov. 1952.

14. G. R. MacLane, *Meromorphic functions with small characteristic and no asymptotic values,* Mich. Math. Jour. *8* (1961) 177-185.

15. G. R. MacLane, *Holomorphic functions, of arbitrarily slow growth, without radial limits,* Mich. Math. Jour. *9* (1962) 21-24.

16. G. R. MacLane, *The geometry of functions holomorphic in the unit circle, of arbitrarily slow growth, which tend to infinity on a sequence of curves approaching the circumference,* Duke Math. Jour. *29* (1962) 191-197.

17. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes,* Paris, Gauthier-Villars 1929.

18. R. Nevanlinna, *Eindeutige analytische Funktionen,* 2e Aufl., Berlin-Göttingen-Heidelberg, Springer Verlag 1953.

19. F. and R. Nevanlinna, *Über die Eigenschaften einer analytischen Funktion in der Umgebung einer singulären Stelle oder Linie,* Acta Soc. Sci. Fenn. *50* (1922) No. 5.

20. I. I. Priwalow, *Randeigenschaften analytischer Funktionen,* Berlin, Deutscher Verl. d. Wiss. 1956.

21. M. Tsuji, *A theorem on the boundary behaviour of a meromorphic function in* $|z| < 1$, Comm. Math. Univ. St. Paul *8* (1960) 53-55.

22. G. Valiron, *Lectures on the general theory of integral functions,* Toulouse, Privat 1923.

23. F. W. Gehring, *The asymptotic values for analytic functions with bounded characteristic,* Quart. Jour. Math. *9* (1958) 282-289

24. M. Heins, *Functions of bounded characteristic and Lindelöfian maps,* Int. Congr. Math. 1958, 376-388.